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volume LIII, *number* 7 November 1960

The transition from school to college mathematics, C. STANLEY OGILVY 514

What mathematics for the terminal student? JACK D. WILSON 518

Problems involving overlapping finite sets, BROTHER U. ALFRED, F.S.C. 524

Right triangle construction, NELSON S. GRAY 533

Remarks on the teaching of calculus in the secondary school, ALBERT A. BLANK 537

An interesting problem involving indeterminate equations, JESSE C. LEWIS 540

Teaching seventh grade mathematics by television, JAMES N. JACOBS *and* JOAN BOLLENBACHER 543

How to make mathematics more attractive, H. M. DADOURIAN 548

Teaching of mathematics in Switzerland, G. PÓLYA 552

Problems in presenting non-Euclidean geometries to high school teachers, LOUIS O. KATTSOFF 559

Is solid geometry an entrance requirement for engineering schools? VINCENT BRANT 564

DEPARTMENTS

Experimental programs, EUGENE NICHOLS 571

Historically speaking,—HOWARD EVES 576

Reviews and evaluations, KENNETH B. HENDERSON 584

Tips for beginners, JOSEPH N. PAYNE *and* WILLIAM C. LOWRY 589

Have you read? 517, 542; *What's new?* 563, 591

THE NATIONAL COUNCIL OF TEACHERS OF MATHEMATICS

Report of the Membership Committee 592

Notes from the Washington Office 597

Your professional dates 599

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The transition from school to college mathematics

C. STANLEY OGILVY, *Hamilton College, Clinton, New York.*

"How can I inject more inspiration and interest into the topics already at hand?'

IT HAS BEEN fascinating and instructive to observe the recent trend in opinion concerning the reform of mathematics teaching in the schools. After twenty years of propaganda in favor of the introduction of the New Mathematics we can now discern the beginning of a swing in the other direction. In almost every new issue of THE MATHEMATICS TEACHER and the *American Mathematical Monthly* we find one or two articles cautioning us to move ahead slowly, to guard against discarding good and valuable old material merely to make room for something new for the sake of its newness. There are even those so bold as to suggest that perhaps, instead of trying to introduce much new mathematics, we should do better to concentrate on revising and revitalizing the old. To put it another way, is it possible to change into real mathematics what has, in some quarters, degenerated into meaningless drudgery and routine?

From this standpoint, the problem undergoes a reformulation. It begins to appear that "How can I introduce more inspiring and interesting new topics into my courses?" was the wrong question. Rather we should have been asking, as the best teachers knew all along, "How can I inject more inspiration and interest into the topics already at hand?" Ultimately the inspiration always comes from the teacher, not from the things that are taught.

It is the aim of summer schools, graduate courses and the National Science Foundation Institutes to restimulate the teacher's interests, to recharge his mathematical batteries. Fortunately the NSF institutes fulfil a dual role. While introducing the teacher to some of the newer fields of modern mathematics, they also restore skills and abilities which may have become dormant through disuse. Thus, whether an institute member returns to new duties or simply to resume the teaching of his former courses, he has acquired a fresh outlook and renewed enthusiasm for his subject, without which successful teaching is almost impossible.

College mathematics instructors are among those who are happy to see the traditional material of high school mathematics emerging from the doghouse. Especially do we rejoice to observe a new awareness that all was not well with the old way of doing things and that some revisions are in order. To be sure, there is not full agreement as to what should be done. Probably this is inevitable: there is no one ideal mathematics curriculum and we may always have to compromise. But progress is definitely being made.

All kinds of radical programs are offered, tried, rejected, revised and tried again. Meanwhile, fortunately, much traditional mathematics continues to be taught. I do not intend to sponsor here a "new program"; you have seen enough of those. Rather I should like to make a few suggestions on a much more modest level. Under the assumption that, whatever else you do, you are going to continue to teach

a lot of algebra, geometry and trigonometry (at least I devoutly hope you will), what, specifically, can you do to present these subjects in a better way? You are continually being exhorted to "improve your teaching"; but seldom does anybody explain how.

I shall mention only one small aspect of the problem, that aspect which concerns me directly. What can you do to improve the mathematical abilities, as viewed and judged by the college teacher, of the students you send to us?

Perhaps I should mention here that, as a former secondary school teacher, I know very well what you are up against. I speak from no ivory tower. Your problems are our problems. If we, as college teachers, did not run into the same difficulties that you do, I should not be writing these words. I am merely trying to enumerate some of the things we should all be thinking about. It is only through close cooperation between school and college that we can get our young people properly started in their mathematical and scientific careers. If I am critical, I criticize myself as much as anyone else. I am reminded of the irate teacher of second year algebra who, losing patience with a slow student, stormed, "Who *was* the dunderhead who let you pass first year algebra? Oh, to be sure—hem—yes, I remember now; it was I."

The incoming freshman receives a big shock when he starts his college mathematics. It takes him a long time to adjust to the great change in level. This change is not so much in course content as in intellectual approach: suddenly much more is expected of him than ever before. Again and again I have freshmen reproachfully say to me, "I can't understand why I am getting C's (or D's or F's); I always got straight A's in high school." The unspoken implication is obvious: "If only you were as good as my high school teacher I might learn something." Usually further questioning reveals that what they were doing in high school was not mathematics at all, but simply some routine processes of solving stereotyped exercises. After the first hour-test in the fall comes the general wail, "But we haven't covered that problem!" That of course is precisely the reason it was on the test. Apparently many high school tests consist only of "policing" questions, checking whether or not the students have "learned" (memorized?) certain standard procedures. These students are seriously shaken to discover that something more than this is required of them if they are to do well in college mathematics.

The kind of question that seems somewhat foreign to their experience, but is the only kind really worth asking, involves applications of known routines to unfamiliar situations. A common complaint is, "I could have solved the problem if I had just known how to start." This confesses a misunderstanding of the facts of life: the problem *was* "how to start." For example, after some plane areas have been found by the standard methods of integration, a test question might hinge on setting two areas equal to each other with the unknown appearing in one of the limits on an integral sign. This of course they would have to set up as well as solve —without ever having done either before. Or perhaps a question might be asked involving an understanding of theory. After a study of the "delta process" of differentiation, one could ask, "Would a negative Δx make any sense? Explain." Obviously this sort of question is worlds away from finding the derivative of a polynomial, expertness in which might delude a high school student into thinking himself good at mathematics.

Needless to say, not all the questions on a freshman calculus test are of this caliber. If they were, I suppose everybody would flunk. But there is a strong shift of emphasis away from drill questions. Method is what counts. One deplores lack of accuracy in numerical work, to be sure, but one does not go into a rage about it. Everybody makes careless errors. The as-

sumption is that, given time and an atmosphere more relaxed than that of an examination room, careless errors could be eliminated. What matters much more is whether the student has the faintest idea of how to go about the problem. What does the question mean? In what general direction lies the answer? What sort of reasoning is involved? What, if any, complicated tools are needed? It is clear that many freshmen have never had to frame these questions. Every "problem" presented in school was in reality no problem at all; it fitted comfortably into a readily recognized groove, and a turning of the crank ground out an answer.

I once remarked to a freshman class that in this course we didn't necessarily get answers; I admitted that we often discussed questions and sometimes derived results. Thereupon one student, who had always labelled the last line "Answer," dutifully changed the label and wrote "Result" at the end of every problem for the rest of the year.

It is regrettable that some high school teachers (surely no readers of THE MATHEMATICS TEACHER) succeed in leaving their students with the impression that mathematics consists of a series of operations in a specified order; that there is only one right way to do anything; and that if one merely follows the leader—I mean the teacher—the answer will appear. This is about as far from real mathematics as it is possible to get; yet some students whose knowledge extends no further made "A" records in high school.

To come down to cases, how many high school boys and girls know why one should "rationalize the denominator" in an "answer"? Why isn't $1/\sqrt{2}$ just as good as $\sqrt{2}/2$? Do *you* know? Your students don't, because every year I ask them and in general I draw a complete blank. The "reason" is supposed to be, of course, that $\sqrt{2}/2$ can be decimalized by short division, once $\sqrt{2}$ has been decimalized, whereas $1/\sqrt{2}$ requires a long division. It ought to be clear, then, that as long as we are not going to decimalize this quantity (and we rarely have any occasion to do so), we might just as well—in fact better—leave it $1/\sqrt{2}$. A student for whom rationalizing the denominator has become a sacred rite is quite blind to the possibility of rationalizing the *numerator*, a maneuver never mentioned in elementary algebra but just as important as the other one. For example, $\sin\theta=\sqrt{2}/2$ may not ring a bell, whereas $\sin\theta=1/\sqrt{2}$ is an old friend.

Another example: $\sin\theta/2=\sqrt{(1-\cos\theta)/2}$ may be well ingrained in a student's memory; yet when confronted with the equivalent identity, $\sin^2\theta=(1-\cos 2\theta)/2$, so important in integration, he says flatly, "We never had that one."

Another: it is easy to differentiate on sight the expression $x-1/x$ if only you leave it alone. But the eager algebraist has been taught that $(x^2-1)/x$ is "simpler form" (don't ask me why). Therefore he must needs write it so, thus making the differentiation more difficult and substantially increasing the chance for error. I am told that the quaint old expression "Reduce your answer to simplest form" is finally falling into disuse. It's about time.

I have had freshmen who thought that $.79788\cdots$ was better than $\sqrt{2/\pi}$, and who actually took the time to change the second into the first by long division and root extraction without even a slide rule! If you want to have some fun, ask the members of your class to write down the most accurate expression they know for the ratio of the circumference of a circle to its diameter. You will get everything from 3 to 22/7, with a majority vote in favor of 3.14159 . . . . Only a very few will think of putting down two vertical straight bars with a horizontal wavy bar across the top.

One could multiply examples, but the point is surely obvious: a technical knowledge is not to be neglected; but if, in addition, a student can acquire a feeling for flexibility, for adaptation, for making the techniques work for him and not against

him, he is on his way toward becoming a mathematician. He must maintain an open mind, be on the alert for possibilities, exploit all known methods, and then be ready to develop new methods if the old ones are inadequate. This is a way of looking at mathematics which does not occur to most beginners.

I close with a humble illustration from my own home. A partition wall was being removed, and a heat outlet would obviously interfere with a new door which had to be installed. There was no way to relocate the door. The heating engineer immediately attacked what to him was a simple problem: "How can we best move the heater out of the way?" His solution involved a rather extensive rebuilding of the ducts in the cellar to move the outlet farther down the wall. With all due respect to a profession which numbers some exceedingly keen experts among its members, I suggest that he was thinking like an engineer. The mathematician on the scene reframed the problem: "How can we best reorganize the system so that the heater will no longer be in the way?" This may sound like the same question, but it is not. The existing heater was reversed, to heat the room on the other side of the wall, and its mate, farther down the wall, was also reversed to take its place. No ducts had to be rebuilt and the job was quickly accomplished, the engineer muttering, "Why didn't I think of that?" You may say that this was simply a matter of common sense, which anyone should have figured out. Of course anyone should; but anyone didn't, which is the only point of the story. Besides, good mathematics properly presented is nothing but advanced common sense.

A student who successfully completes a college mathematics major has learned far more than a new collection of techniques. We hope that he has begun to learn how to think like a mathematician. This means that he has acquired not only the knack of solving problems, although that is important; but also the ability to *formulate* the problem, to get it ready to solve. The man who is able to formulate problems, to analyze them and arrange them so that they become capable of solution, is in the highest demand today. If we can succeed in helping our youngsters to acquire this ability through their mathematics courses, we shall have made them more useful to society, to their employers, and to themselves.

Have you read?

BELLMAN, RICHARD, and BROCK, PAUL. "On the Concepts of a Problem and Problem-Solving," *The American Mathematical Monthly*, February 1960, pp. 119–134.

Problem solving is always an interesting subject, and there will always be opportunities to increase our efficiency in this activity. Although this article is on computing devices, it seems to me it has implications for all of us.

The author divides problems into three types: natural, model, and symbolic. He points out that the concept of what constitutes a solution is changing, and that more and more emphasis is placed on the solution process. The solution process asks if reductions are necessary, if so how reduced, if the reduced problem has a solution, and, if not, can it be reduced again? This can be a never-ending process. Each type of problem has its particular solution process. The solution to a symbolic problem must be ideal and contain three difficulties: analytic, computational, and conceptual.

Computers find the conceptual difficulty most troublesome. Your students will be interested in the illustrations given and how some simple ones give rise to conceptual conundrums. If you never see a computer, this discussion should help you in problem solving.—PHILIP PEAK, *Indiana University, Bloomington, Indiana.*

What mathematics for the terminal student?

JACK D. WILSON, *San Francisco State College, San Francisco, California.*

A question that still needs an answer even though one has been sought for years.

MATHEMATICIANS and teachers of mathematics in many schools and colleges have recently made the decision to reform the old high school mathematics curriculum and to install a new modernized program. Efforts in this direction have not been successful in the past, but today four new ingredients have been added:

1. Unlimited amounts of money, running into millions of dollars, are available to those who are organizing the new programs.
2. University mathematicians are actively co-operating with high school teachers in the planning and writing of new materials.
3. The public is more sympathetic to mathematics and to mathematics teachers than at any time in the past.
4. And, finally, classroom teachers are themselves exhibiting a degree of enthusiasm far beyond anything seen in earlier periods. (Compare, for instance, the registration of over 800 at a recent two-day California Asilomar conference with the enrollments of 150 to 200 which were common only three or four years ago.)

In this first great surge of excitement, a great deal of energy has been expended in an effort to provide new courses for the college preparatory student, but very little attention has been devoted to the needs of other types of pupils, especially to the so-called terminal students. So we rightly ask: What kind of mathematics is most appropriate for these students? What course content is most promising? What kinds of teachers are needed? And, finally, what role should the teacher-education institution play in the preparation of teachers for the terminal courses?

Now, I think it is self-evident that we cannot solve this problem in isolation. We must look at the over-all picture, past and present. We need answers to the questions:

1. What is going on in the elementary grades? What are the implications of these developments for the secondary program?
2. What efforts have been made in the past in the area of mathematics for general education? Do these efforts offer any clues for the future?
3. To what extent should mathematics for the terminal student be the same, to what extent should it be different from that offered in the college preparatory sequence?

Let us turn our attention to the first of these questions: What is going on in the elementary grades?

Even a casual survey shows some very interesting developments. As you know, starting about thirty-five years ago, arithmetic was gradually pushed up through the grades, with some schools even eliminating the subject entirely from the first three years. This was called the stepped-up curriculum. And one superintendent

stepped arithmetic up so far that no formal arithmetic at all was taught in the first five or six grades. The resulting situation was depicted in cartoon form as follows: A class of pupils are all busily engaged with sandbox toys, except for one little boy who is standing at the chalkboard on which he has just written "2+2= "; a somewhat impatient teacher is standing over him and saying, "Come, come, Jimmie, stop fooling around."

The results of this unfortunate movement were well-nigh disastrous, and we are still feeling its effects. Of course, the arithmetic people eventually recognized the folly of their postponement program, and now arithmetic has been pushed down again. Today it is almost back to where it was thirty years ago.

And naturally this movement has important implications for the secondary school mathematics program of our terminal students. Presumably they will reach the high school level with a better arithmetic background than heretofore, if for no other reason than a prolonged exposure to the subject.

But there are other changes. Thirty years ago, arithmetic was primarily a drill subject, dull, dry, and unpopular. As a reaction to this very unpleasant arithmetic, there began a strong movement to socialize the subject, to build a "daily life" arithmetic. This was the heyday of the theory of social utility, the period when Professor Guy Wilson and his co-workers were carrying out extensive experiments to determine what mathematics adults actually used and hence what mathematics should be taught to children in the grades.

But this extreme socializing of the subject failed. It failed because it did not take sufficient cognizance of the fact that arithmetic has a systematic logical structure. To understand arithmetic and use it properly, one must discover the meanings underlying the fundamental processes. These meanings are not to be found in social situations; they are only to be found in the number system itself.

Thus from a feeble start in 1930, the drive to make arithmetic meaningful gathered strength during the forties and fifties. You all know the leaders in this movement—Brownell, Buswell, Grossnickle, Brueckner, Spitzer, Carpenter, Hartung, Van Engen, and many others who have worked long and hard toward their goal. And the end is not in sight! Just recently there have been new and novel efforts in this direction with some textbooks introducing variables and ratios in the fourth grade and systematically building understanding of these key concepts in the fifth and sixth grades. Thus in one textbook series, all three cases of per cent are solved via the use of ratios; the student follows a systematic plan of attack, always setting up two equal ratios regardless of the type of problem with which he is confronted. Such an approach, based on patterns and structure, promises to take some of the mystery out of per cent, especially Case III which has long been recognized as very difficult for the average child.

Solving problems through simple equations is not a strange or radical innovation. It is an evolutionary movement arising out of the work of the past twenty years and, as such, is a natural development. However, it does not go far enough to satisfy some of the persons who have recently become interested in the elementary curriculum. I refer, for instance, to Suppes and Hawley who are introducing geometry in Grades 1 and 2; to David Page and others who are experimenting with various teaching methods and mathematical ideas at the University of Illinois; and to the School Mathematics Study Group which has published several extremely interesting units for use in Grades 7 and 8. These represent only a very incomplete listing of persons who have recently turned their attention to the elementary mathematics curriculum.

And from the combined efforts of all these people, there will certainly come a better elementary curriculum than we have today, although hazards do exist,

especially if premature action presents us with a pure science of number taught with little regard for the social purposes.

But whatever happens, one conclusion is inescapable: much more arithmetic will be taught in the lower grades, and what is taught will be presented with increased emphasis on mathematical structure. And this, in turn, will exert an all-important influence on the kind and amount of mathematics which is teachable in the upper grades. Thus, in selecting learning materials for terminal students in our high school grades, we cannot afford to ignore the newer developments in the elementary school; we must exercise care to choose topics which grow naturally out of these newer experiences.

Let us now turn our attention to the second series of questions: What kind of mathematics has been provided for terminal students in the past? Does past experience offer any clues for the future?

Now we all know that teachers have been wrestling with the problems of terminal mathematics for at least thirty years. Articles have been written; courses of study have been worked out; textbooks have been published; committees have reported. Yet we are still looking for answers.

In the thirties, we tried general mathematics—an integration of old-fashioned algebra and geometry with some arithmetic and trigonometry. But this effort did not meet with favor; the climate of opinion was just not right. In those depression days of the thirties, we could not justify mathematics "for its own sake"; we had to show that the subject was useful in daily-life activities. And this the early courses in general mathematics did not do.

As a matter of fact, this point of view prevailed not only in general mathematics but even in geometry. We are all familiar with the attempts to teach geometry in such a way that the methods learned would carry over into ordinary life situations. The most famous effort in this direction was the "Nature of Proof" yearbook by Harold Fawcett.

Thus during the late thirties and early forties there developed a socialization movement in all our mathematics courses, but especially in general mathematics. Many persons, including the authoritative Commission on Post-War Plans, recommended the establishment of multiple tracks in secondary mathematics, sometimes two, sometimes three or more—one track for the upper 15 per cent and other tracks for the remaining 85 per cent.

All through this period the general courses received much more attention than did their traditional rivals, algebra, geometry, and trigonometry. Over and over again these questions were asked: What mathematics is needed for citizenship? What mathematics for everyday occupations? What mathematics for intelligent reading of newspapers and magazines? But most of all, what mathematics is needed in the home?

As this questioning continued, there developed a strong movement for a general mathematics which differed from the traditional courses not only in *degree* but also in *kind*, a general mathematics which placed primary emphasis on skills and social applications.

Parenthetically, it must be admitted that this general mathematics frequently degenerated into remedial arithmetic, but it must also be granted that many schools did much better.

Now what did the average teacher think of this attempt to socialize the subject matter of general mathematics? This is a difficult question to answer and must be viewed in the light of the conservatism of the average teacher. So slow have teachers been to adopt new ideas in the past that it has usually taken about twenty years for a new movement to gain general acceptance. One explanation for this reluctance to adopt new materials is that the risks have seemed large and the rewards mainly hard work. Nevertheless, in this instance, teachers were willing to make a serious and immediate effort to implement a program of social mathematics.

As we follow the march of events into the fifties, we find mathematics teachers, individually and in committee, waging a continuing struggle to organize mathematics within socially-significant situations. In California, the major study group was the state-wide Committee for the Study of Education which brought in its report in January, 1957.

And what recommendations did this committee have for the terminal student?

I think it would be fair to say that there was a continuing emphasis on socialized mathematics with particular attention to the problems of housekeeping and business. Four courses were suggested:

Mathematics 9: Mathematics of General Education
Mathematics 10: Mathematics of the Industries of the Community
Mathematics 11: Mathematics of the Businesses of the Community
Mathematics 12: Mathematics of the Consumer-Citizen.

Two other recommendations indicate the flavor of this report:

1. The suggestion that college faculties should work more closely with business and industry so as to understand better the need for a broad training of mathematics teachers; and
2. The recommendation that future mathematics teachers should take an introductory survey course covering the quantitative aspects of economic, social and political problems of the consumer.

In reading this report, one conclusion is inescapable: the problem of providing suitable mathematics experiences for the terminal student had not yet been solved. And we naturally wonder, after all these years of effort, why we seem to be so far from the solution. With all the thinking that has gone into the organization of consumer mathematics courses, to mathematics for everyday life, why are we so far from our goal? I am going to suggest four reasons.

1. In my opinion, courses in consumer mathematics do include many socially useful topics, but these topics do not involve enough real mathematics. For instance, abilities such as filling out income tax forms and planning personal budgets are important social competencies, but they are not really mathematics.
2. There is reason to believe that high school students find these social units neither interesting nor important. For example, the study of mortgages, insurance, and taxes has little realism for a sixteen-year-old who will not buy his own home until many years have passed.
3. High school teachers, burdened by responsibilities, just cannot spend a lot of time searching for new socialized materials, especially today when they are obligated to learn and to teach so much new mathematics.
4. Finally, and most important, we are living in a period of such rapid change that we cannot possibly plan an educational program on the basis of social utility. We cannot possibly predict the social needs of twenty, ten, or even five years hence; consequently, it is extremely hazardous to organize a program on the basis of usefulness in everyday life.

We have no alternative, therefore, but to turn to mathematics itself for our source of teaching materials. We do have a pretty good idea as to what concepts will be important in the future, even though we cannot know with any degree of certainty what skills will be needed or what social applications will be valuable. Further, I think we would all agree that a student who understands the concepts of mathematics will display a greater versatility in handling quantitative problems in later life than will a student whose instruction has been rooted mainly in social

applications. Thus the teaching of concepts must occupy the central position in all our high school mathematics courses.

This brings us to a consideration of the third question that we posed at the beginning of this paper: In what way should mathematics for the terminal student be the same, in what way should it be different from the mathematics in the college preparatory sequence.

I think the answer to this question is evident from the foregoing discussion. Regardless of the nature of the student body, course content must be focused on mathematical principles; mathematics for the college preparatory student and mathematics for the terminal student will differ only in degree.

But this decision carries with it an important responsibility. We must insure that the course content for our terminal students does not overemphasize concept building to the point where skills and applications are neglected. In this connection, there is the story of a teacher writing simple arithmetic exercises on the blackboard and a bright-looking little fellow whispering to another with great seriousness, "It isn't the atomic theory that bothers me—it's the adding and subtracting." In the present context, this same boy might be saying, "I know the postulates of a field but I can't add or subtract." Or, "Yes, I understand the distributive law but I can't multiply or divide."

Thus, for the terminal student, we must proceed slowly in our enthusiasm for generalizations, in our desire to achieve a high level of abstraction. We must not lose sight of the fact that many high school students find abstractions difficult, that proofs acceptable to mathematicians leave the average student completely cold. We will need to make a careful selection of the concepts and ideas to be developed if we are not to defeat our main purpose. And, as we teach each concept, we must always try to show some relationship with ordinary life activities, even if these relationships are somewhat fanciful and artificial. However, you will notice that this approach differs from that employed in social mathematics in a fundamental way: the mathematical concepts become central to our study; the applications are peripheral and are introduced primarily to motivate the mathematics. In other words, we are not so much interested in a specific bit of knowledge as in ability to think in mathematical terms.

Now it would go far beyond the scope of this paper to try to outline a mathematics course for terminal students; in fact, much experimentation may be necessary in order to develop a program which will be even partially successful. The most we can do here is to list a few topics which appear to qualify as good vehicles for developing understanding of mathematical concepts and which, at the same time, are sufficiently concrete to hold the interest of the terminal student. We will comment briefly on each of the following: numeral systems, structure of algebraic systems, elements of statistics, elements of algebra, and review of basic fundamentals.

1. *Numeral Systems.* The study of numeral systems with bases other than ten helps to develop a broader understanding of our own system, especially if presented in an interesting quasi-historical fashion. Further, the instructor can show how other numeral systems, such as the binary, are employed in modern business applications. But these specific applications are merely convenient sources of motivation to reinforce the mathematical concepts.

2. *Structure of Algebraic Systems.* An attempt must be made to give the terminal student some idea as to the nature and structure of algebraic systems. We can, for instance, use a five-hour clock to build addition and multiplication tables, modulo five, and, from the tables, illustrate closure, the associative law, the existence of identity elements, the existence of inverse elements, and the commutative law. This

activity should help to clarify the student's understanding of the number systems that he meets in ordinary life: the whole numbers, the integers, the rationals and the reals.

3. *Elements of Statistics.* All our students should have some instruction in elementary statistics but again the development of concepts must occupy a central position. For example, we can show the need for precise definitions by comparing the vagueness of "average" with the exact meanings assigned to "mean" and to "median." Similarly, we can illustrate the role played by assumptions even in the relatively simple problem of computing the mean from grouped data: the student must assume that the scores in any interval are uniformly distributed throughout that interval; then, in computing the mean, he can assume that all scores in a given interval can be considered as located at the mid-point of that interval.

4. *Elements of Algebra.* The elements of algebra (modern approach) should be presented and developed to whatever degree is feasible in a given classroom. Various practical applications involving formulas and ratios will provide a link with life situations. A culminating activity might be the development and utilization of the constant ratio formula for computing interest rates in installment buying.

5. *Review of Basic Fundamentals.* In courses for terminal students, we must arrange for interesting drill exercises (magic squares, tack-on and teens multiplication, early methods of multiplication, and various puzzles and games) as well as diagnostic and remedial instruction and straightforward basic drill.

These suggestions represent only a few of the many possibilities that are available to those who are organizing courses for terminal students.

Finally we come to this question: What role should the teacher-education institution play in the preparation of teachers for terminal mathematics courses? On the basis of our earlier discussion, I think we can say with some confidence that the training for teaching terminal mathematics will not require any special courses. The new requirements for the mathematics-teaching major already include courses like modern algebra, modernized college geometry, statistics, probability, and arithmetic from an advanced viewpoint, together with a professionalized curriculum and instruction course to pull together the ideas developed in other courses and to consider how they can be best utilized in the mathematics program of the terminal student. And this should be adequate!

The road ahead is clear. Never before have there been so many factors favorable to our cause. Never before have we had such a golden opportunity to build a strong mathematics program. We must take advantage of this unparalleled opportunity, but to do so we must be nimble and we must be quick. As rapidly as possible, we must place our mathematics program on a foundation so firm that its future will be assured for our time and the time of our children.

And the stakes are high! A country lacking a good mathematics program cannot long maintain a strong position in the modern world of advanced technology.

Many small schools are now encouraging capable pupils to progress more rapidly than normal by taking correspondence or extension courses in subjects not otherwise available.—"Schools in Our Democracy," U.S. Department of Health, Education, and Welfare, Office of Education.

Problems involving overlapping finite sets

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In a sampling of 1,000 persons it was found that 215 took at least two of four leading magazines, 35 took all four, while 125 took none. A total of 965 copies was involved. Are these data consistent?

IN THE ATTENTION that has come to be given to sets at the secondary level, it is usual to find passing reference to problems such as the following:

> In a group of black and white rabbits, 85 are wholly or partly white, while 67 are wholly or partly black. If there are 137 rabbits in all, how many are all white, how many all black, how many partly white and partly black?

This is the very simplest type of problem involving overlapping finite sets.

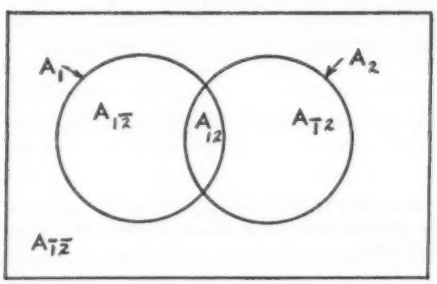
For the purposes of this article, we shall define the term *set* as any collection of well-defined objects, distinguishable one from another.

The *universe* (under consideration) is a set of objects from which the given set or sets are drawn.

The *intersection* of two sets is the subset consisting of their common elements.

In the foregoing example, we can picture the situation by means of a Venn diagram. The rectangle represents the universe, while the circles represent two subsets, their common area being the intersection subset. (See Fig. 1.)

Figure 1



Let A_1 = the set of white rabbits, wholly or partly white,
A_2 = the set of black rabbits, wholly or partly black,
A_{12} = the set of black-and-white rabbits.

For each set there is a *cardinal number*, that is, the number of objects in the set which by supposition is finite. Corresponding to each of these sets, we have cardinal numbers N_1, N_2, and N_{12} respectively.

In the given figure are shown four main areas:

$A_{\bar{1}\bar{2}}$ indicates the set of rabbits with no white or black.
$A_{1\bar{2}}$ indicates the set of wholly white rabbits.
A_{12} indicates the set of white and black rabbits.
$A_{\bar{1}2}$ indicates the set of wholly black rabbits.

These particular sets may be denoted the *ultimate sets*, as all other sets may be represented as combinations of them. For example,

$$A_1 = A_{12} \cup A_{1\bar{2}}$$

since, logically, white rabbits must be either white and black or white and not black.

This has been a long introduction to solving a very simple problem. We have the following data given:

$N_1 = 85$,

$N_2 = 67$,

$N = 137$ (total number in the universe).

But $N = N_{\bar{1}\bar{2}} + N_{1\bar{2}} + N_{12} + N_{\bar{1}2}$, while
$N_1 = N_{12} + N_{1\bar{2}}$ and $N_2 = N_{12} + N_{\bar{1}2}$.
Hence $N - N_{\bar{1}\bar{2}} = N_1 + N_2 - N_{12}$. Since $N_{\bar{1}\bar{2}}$ is implied to be zero,

$$137 = 85 + 67 - N_{12}$$

or

$N_{12} = 15$, number of black-and-white rabbits;
$N_{1\bar{2}} = N_1 - N_{12} = 85 - 15 = 70$, number of wholly white rabbits;
$N_{\bar{1}2} = N_2 - N_{12} = 67 - 15 = 52$, number of wholly black rabbits.

Such a problem, while instructive in an introductory way, does not have sufficient variety to bring out some of the more interesting features of overlapping finite sets. We proceed, therefore, to a problem involving three different sets which is more or less the ideal case to consider in arriving at a more comprehensive view. Again let us take a problem, such as the following:

For a small town, statistics indicated that of the male adults (7,635 in all)
5,492 were coal miners,
6,057 were Democrats,
4,135 were affiliated with a church,
4,532 were coal miners and Democrats,
3,957 were church affiliates and Democrats,
3,553 were church affiliates and coal miners,
3,337 were church affiliates, Democrats, and miners.
How many were not church affiliates, nor Democrats, nor miners?

Following the previously adopted notation, let A_1 be the set of coal miners, A_2, the set of Democrats, and A_3, the set of church affiliates. The corresponding cardinal numbers would be N_1, N_2 and N_3. Here we have a Venn diagram in which we mark eight different areas. (See Fig. 2.) These correspond to the ultimate sets mentioned above. Thus the set of non-coal miner, non-Democratic, non-church members is indicated as $A_{\bar{1}\bar{2}\bar{3}}$, the set of coal miners who are not Democrats or church members is indicated $A_{1\bar{2}\bar{3}}$, and so on. It will be noted that the ultimate sets involve, in addition to positive characteristics, negative characteristics as well. Thus, $A_{1\bar{2}\bar{3}}$ means the set of men who are coal miners, but not Democrats or church affiliates. On

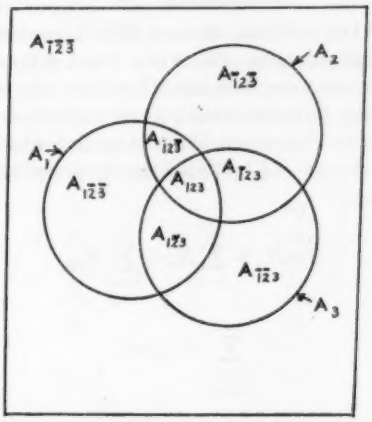


Figure 2

the other hand, the given data involve positive characteristics only, the seven figures corresponding respectively to N_1, N_2, N_3, N_{12}, N_{13}, N_{23}, N_{123}. We are also given N, the total number in the universe. The quantity we are seeking is $N_{\bar{1}\bar{2}\bar{3}}$, the cardinal number of the set outside the three given sets in the Venn diagram.

The problem, therefore, is to find a relation among the positive characteristic frequencies and N and $N_{\bar{1}\bar{2}\bar{3}}$. We can start with the fundamental relation:

$$N = N_{123} + N_{12\bar{3}} + N_{1\bar{2}3} + N_{\bar{1}23} + N_{1\bar{2}\bar{3}} + N_{\bar{1}2\bar{3}} + N_{\bar{1}\bar{2}3} + N_{\bar{1}\bar{2}\bar{3}},$$

which states that the cardinal number of the universe is the sum of all the cardinal numbers of the ultimate sets. There are three pairs, such as $N_{12\bar{3}} + N_{123}$, which can be transformed as follows:

$$N_{12\bar{3}} + N_{123} = N_{13} = N_1 - N_{13}$$

Using these relations, we find that:

$$N = N_{\bar{1}\bar{2}\bar{3}} + N_1 + N_2 + N_3 - N_{12} - N_{13} - N_{23} + N_{123}$$

Substituting the values as given in the problem, we have

$$7{,}635 = N_{\bar{1}\bar{2}\bar{3}} + 5{,}492 + 6{,}057 + 4{,}135 - 4{,}532 - 3{,}957 - 3{,}553 + 3{,}337$$

or $N_{\bar{1}\bar{2}\bar{3}} = 656$.

Generalization

When we pass to more than three sets, visualization by means of a Venn diagram becomes more difficult. However, the following formula results as an extension of the previous cases. First let us indicate by N_0 the set not included in any given sets. Then:

$$N = N_0 + \sum_{i=1}^{n} N_i - \sum_{\substack{i,j=1 \\ i<j}}^{n} N_{ij}$$

$$+ \sum_{\substack{i,j,k=1 \\ i<j<k}}^{n} N_{ijk} - \cdots$$

or, if we use the convenient notation: $S_1 = \sum N_i$; $S_2 = \sum N_{ij}$; etc.,

$$N = N_0 + S_1 - S_2 + S_3 - S_4 \cdots,$$

so that, for n sets,

$$N = N_0 + \sum_{k=1}^{n} (-1)^{k+1} S_k$$

Building up set information from original data

The usual problems about overlapping sets state certain numerical data, such as were given in the two problems above. But experience seems to point to the conclusion that a great deal of insight into the nature of the situation is lost unless we go back to the originally collected data and see precisely how they are related to the statements about sets. From the pedagogical standpoint, it would probably be a good idea to collect some anonymous, minor social statistics in the class, as the number of cases involved is not large. Such a procedure would tend to add interest. Thus, each student might be given a list of four curriculum subjects being studied by the members of the class and asked to check the ones he likes. These data, duly arranged in a table with one line for each student and four columns corresponding to the four subjects, would be the raw material for the investigation of sets.

If it is deemed undesirable to adopt this procedure, one need merely arrange a table

TABLE I

Case	*Sets*					*Total*
	I	*II*	*III*	*IV*	*V*	
1	x		x	x		3
2		x				1
3	x	x		x		3
4				x	x	2
5		x	x			2
6		x		x	x	3
7	x					1
8		x		x		2
9						0
10	x			x		2
11			x			1
12		x		x	x	3
13				x		1
14		x	x		x	3
15	x					1
16			x			1
17		x		x		2
18	x		x		x	3
19						0
20		x	x			2
21	x				x	2
22	x	x		x		3
23			x	x	x	3
24			x			1
25				x		1
26	x	x	x	x	x	5
27		x	x	x	x	4
28	x	x	x	x		4
29		x			x	2
30	x		x			2
31		x	x	x	x	4
32	x		x	x		3
33		x				1
34			x			1
35	x	x	x	x	x	5
36	x					1
37		x				1
38			x			1
39				x		1
40					x	1
41		x	x	x		3
42	x	x	x		x	4
43		x		x		2
44			x			1
45				x	x	2
46	x	x				2
47	x		x	x	x	4
48	x	x		x	x	4
49			x			1
50	x	x		x	x	4
Totals	19	24	23	25	18	109

with columns corresponding in number to the number of sets in question and horizontal lines equivalent to the individual cases. Then by placing checks more or less at random in the spaces, one can create data for the study of sets. Table I represents such an arrangement.

Ultimate class frequencies. A moment of thought will convince one that the ultimate class frequencies may be found directly from the table. For five sets, there are 2^5 or 32 possible ultimate frequencies. Case 1, for example, belongs to set A_{12345}; Case 2, to A_{12345}; and so on.

Inasmuch as listing the ultimate sets and ultimate frequencies corresponding to them in the form A_{12345} or N_{12345}, respectively, involves a great deal of writing, the following convenient notation is adopted in this article. For A_{12345} we shall write $(A_{12})_0$, the subscript zero signifying that only the positive characteristics indicated by the subscripts inside the parentheses are involved and all others excluded. To point up the difference between $(A_{12})_0$ and A_{12} or between $(N_{12})_0$ and N_{12}, it may be observed that A_{12} includes the following ultimate sets: A_{12345}, A_{12345}, A_{12345}, A_{12345}, A_{12345}, A_{12345}, A_{12345}, and A_{12345}, whereas $(A_{12})_0$ means simply A_{12345}.

The summary of ultimate sets (using both notations this time only, to make clear their relations) is given in Table II.

TABLE II

ULTIMATE SET FREQUENCIES FOR THE DATA OF TABLE I

First Notation	*Second Notation*	*Set Frequency*
N_{12345}	N_0	2
N_{12345}	$(N_1)_0$	3
N_{12345}	$(N_2)_0$	3
N_{12345}	$(N_3)_0$	7
N_{12345}	$(N_4)_0$	3
N_{12345}	$(N_5)_0$	1
N_{12345}	$(N_{12})_0$	1
N_{12345}	$(N_{13})_0$	1

And so on.

A method to determine the positive characteristic frequencies $N_1, N_2, N_3, \cdots, N_{12}, N_{13}, N_{14}, \cdots$, to N_{12345}, applicable to a small number of cases, is direct counting. First of all, by adding the number of checks in each column, of Table I, we have the following frequencies:

N_1	19
N_2	24
N_3	23
N_4	25
N_5	18
Total	109

This total should agree with the sum of the figures in the totals column at the right.

To find N_{12}, we go down the columns and determine in how many instances sets 1 and 2 are found together; similarly for N_{13}, N_{14}, and the rest. It is not difficult to see that this method holds the possibility of error. Therefore, some form of check is desirable.

The procedure can be illustrated most simply by an example. From the figures in the totals column giving the number of sets in which each case is involved, we formulate Table III showing the frequencies associated with a given number of sets.

TABLE III

No. of Sets	*Frequencies*
0	2
1	17
2	12
3	10
4	7
5	2
TOTAL	50

Also from Table 1, the frequencies of the sets of the second order (i.e., those involving two positive characteristics) are found to be:

N_{12}	9	N_{23}	10	N_{35}	9
N_{13}	9	N_{24}	15	N_{45}	12
N_{14}	11	N_{25}	11		
N_{15}	8	N_{34}	10		

The sum of these figures is $S_2 = \sum N_{ij} = 104$. Now, consider a case involving 4 sets; such a case generates six of the above frequencies inasmuch as there are six ways in which four objects can be selected two at a time. Therefore the sum of second order frequencies on page 527 should equal the following:

$$\begin{aligned} 2 \quad {}_5C_2 &= 20 \\ 7 \quad {}_4C_2 &= 42 \\ 10 \quad {}_3C_2 &= 30 \\ 12 \quad {}_2C_2 &= 12 \\ &\overline{104} \end{aligned}$$

where ${}_nC_r$ represents the number of combinations of n things taken r at a time.

The remaining positive characteristic frequencies of the third, fourth, and fifth orders are found to be:

N_{123} 4	N_{145} 5	N_{1234} 3	N_{12345} 2
N_{124} 7	N_{234} 6	N_{1235} 3	
N_{125} 5	N_{235} 6	N_{1245} 4	
N_{134} 6	N_{245} 8	N_{1345} 3	
N_{135} 5	N_{345} 6	N_{2345} 4	

and $N_0 = 2$. This last is the completely negative characteristic frequency, but is associated with the positive characteristic frequencies to form a complete set suitable for making calculations.

Summing the positive characteristic frequencies of the various orders, we have:

$$\begin{aligned} S_1 &= \sum N_i = 109 \\ S_2 &= \sum N_{ij} = 104 \\ S_3 &= \sum N_{ijk} = 58 \\ S_4 &= \sum N_{ijkl} = 17 \\ S_5 &= \sum N_{ijklm} = 2 \end{aligned}$$

As an additional check, N should equal (see above):

$$\begin{aligned} N_0 + S_1 - S_2 + S_3 - S_4 + S_5 \\ = 2 + 109 - 104 + 58 - 17 + 2 = 50 \end{aligned}$$

Positive characteristic frequencies from ultimate set frequencies. When the number of items is large, a more practical and much quicker method is to find the ultimate set frequencies first and from these determine the positive characteristic frequencies. The procedure is illustrated for artificially prepared data involving four sets and 150 cases, shown in Table IV.

TABLE IV

ULTIMATE SET FREQUENCIES FOR 150 ITEMS

N_0	19	$(N_{23})_0$	15
$(N_1)_0$	9	$(N_{24})_0$	10
$(N_2)_0$	12	$(N_{34})_0$	8
$(N_3)_0$	17	$(N_{123})_0$	8
$(N_4)_0$	4	$(N_{124})_0$	8
$(N_{12})_0$	12	$(N_{134})_0$	3
$(N_{13})_0$	11	$(N_{234})_0$	6
$(N_{14})_0$	4	$(N_{1234})_0$	4
			150

The process of determining the positive characteristic frequencies will be illustrated for first order frequencies. To find N_1, for example, you simply add all the frequencies for the ultimate sets involving the subscript 1. Thus,

$$N_1 = 9+12+11+4+8+8+3+4 = 59$$

$$N_2 = 12+12+15+10+8+8+6+4 = 75$$

$$N_3 = 17+11+15+8+8+3+6+4 = 72$$

$$N_4 = 4+4+10+8+8+3+6+4 = 47$$

Once again, a check can be made by finding the sum of these first order frequencies, 253. This should equal:

$$\begin{aligned} 4 \quad {}_4C_1 &= 16 \\ 25 \quad {}_3C_1 &= 75 \\ 60 \quad {}_2C_1 &= 120 \\ 42 \quad {}_1C_1 &= 42 \\ &\overline{253} \end{aligned}$$

The same procedure can be used to determine frequencies of other orders. For example, to find N_{12}, simply add the frequencies of all ultimate sets involving these subscripts. Thus,

$$N_{12} = 12+8+8+4 = 32$$

FORMULATING SET PROBLEMS

As should be apparent, it is not possible to list any numbers whatsoever as positive

characteristic frequencies and expect them to be consistent with a given array of ultimate set frequencies. For the teacher interested in making up set problems, a simple method can be found in the work that has just been carried through. It comes to this:

1. Any numbers can be set down as ultimate set frequencies.
2. We can then calculate the positive characteristic frequencies from these ultimate set frequencies.

Ultimate set frequencies from positive characteristic frequencies. To determine the ultimate set frequencies from the positive characteristic frequencies and N_0, the frequency of the set with completely negative characteristics, we need only reverse what was done in the previous section. First we observe that for the highest order given, the positive characteristic frequencies correspond to the ultimate set frequencies. To find a frequency of any lower order, we subtract from the positive characteristic frequency in question the ultimate set frequencies of higher order containing the same subscripts.

The process is illustrated in the following example:

Frequency Symbol	Positive Characteristic Frequency	Deductions	Ultimate Set Frequencies
N_{123}	6		6
N_{124}	8		8
N_{134}	7		7
N_{234}	2		2
N_{12}	22	−6−8	8
N_{13}	25	−6−7	12
N_{14}	28	−8−7	13
N_{23}	23	−6−2	15
N_{24}	19	−8−2	9
N_{34}	13	−7−2	4
N_1	68	−6−8−7−8−12−13	14
N_2	70	−6−8−2−8−15−9	22
N_3	62	−6−7−2−12−15−4	16
N_4	53	−8−7−2−13−9−4	10
N_0	25		25

Checking consistency in set data

To determine whether set data are consistent, one straightforward method is to work back to the ultimate set frequencies. As was pointed out previously, these figures are equivalent to the original information from which the set numbers were derived. It is obvious that each must be greater than or equal to zero. If any turn out to be negative, the data must be inconsistent.

Where the information is given in such form that it is not possible to go back to the ultimate set frequencies, we calculate the quantities S_i. Should any of these be negative, there is a lack of consistency in the data.

And of course, the value for N as determined by the formula $N=N_0+S_1-S_2+S_3-S_4\cdots$ should agree with the given value.

In the event that we are not interested in obtaining all the original data or ultimate set frequencies, but only some of them, we use special formulas suitable for our purpose. For example:

$$(N_{12})_0=N_{12}-\sum_i N_{12i}+\sum_{i,j} N_{12ij}-\sum_{i,j,k} N_{12ijk}+\cdots,$$

an ultimate set frequency in terms of the positive characteristic frequencies. For the 50-item example in Table I, we have:

$$(N_{12})_0=9-(4+7+5)+(3+3+4)-2=1$$

which agrees with the ultimate set frequency for this case (Case 46). It will be seen that this formula bears a strong resemblance to the general formula

$$N=N_0+\sum N_i-\sum N_{ij}+\sum N_{ijk}\cdots$$

Other problems

Problem. The 762 students in a given high school participated in extracurricular activities as follows:

(1) Dramatics	25
(2) Journalism	20
(3) Band	45
(4) Football	48
(5) Glee Club	40

There was a ruling in the school that no student might participate in more than three such activities. The following array indicates the combinations of activities and their frequencies:

$N_{12}=5$ $N_{23}=4$ $N_{45}=5$ $N_{135}=2$
$N_{13}=6$ $N_{24}=3$ $N_{145}=2$
$N_{14}=3$ $N_{25}=1$ $N_{123}=3$ $N_{235}=1$
$N_{15}=10$ $N_{35}=3$ $N_{124}=2$

For combinations not listed the frequencies are zero.

1. How many students in the school did not participate in any of these activities?
2. How many had exactly one activity?
3. How many had exactly two activities?

Problem. A club consisted of 235 people. It was found that 125 subscribed to *Life*, 87 to the *Reader's Digest* and 75 to *Time*. A total of 45 subscribed to none of these publications, while 76 subscribed to at least two of them. How many subscribed to all three?

These and similar problems involve such ideas as "at least two," "at most two," or "exactly two." Such information can often be found by the methods previously discussed, such as reconstructing the original data or what comes to the same thing, calculating the ultimate set frequencies. However, this is not always the most convenient or easiest procedure.

For convenience in handling these ideas, we let

$N_{\text{II}+}=$ number of elements in at least two sets,

$(N_{\text{II}})_0=$ number of elements in exactly two sets.

Some general relations among these quantities are immediately evident, such as:

$$N_{\text{II}+}-N_{\text{III}+}=(N_{\text{II}})_0$$

and

$$N=N_0+(N_{\text{I}})_0+(N_{\text{II}})_0+(N_{\text{III}})_0\cdots+(N_n)_0$$

It is also evident that

$$\begin{aligned}N_{\text{I}+}&=N-N_0\\&=S_1-S_2+S_3-S_4\cdots+(-1)^{n-1}S_n\\&=\sum_{k=1}^{n}(-1)^{k+1}S_k\end{aligned}$$

The following relations may be derived:

$$\begin{aligned}N_{\text{II}+}&=S_2-2S_3+3S_4-4S_5+\cdots\\&=\sum_{k=2}^{n}(-1)^k\,{}_{k-1}C_1S_k\end{aligned}$$

$$\begin{aligned}N_{\text{III}+}&=S_3-3S_4+6S_5-10S_6+\cdots\\&=\sum_{k=3}^{n}(-1)^{k+1}\,{}_{k-1}C_2S_k\end{aligned}$$

$$\begin{aligned}N_{\text{IV}+}&=S_4-4S_5+10S_6-20S_7+\cdots\\&=\sum_{k=4}^{n}(-1)^k\,{}_{k-1}C_3S_k\end{aligned}$$

$\cdots$

In general,

$$\begin{aligned}(N_R)_+&=S_R-{}_RC_{R-1}S_{R+1}+{}_{R+1}C_{R-1}S_{R+2}\\&\quad-{}_{R+2}C_{R-1}S_{R+3}+\cdots\\&=\sum_{k=R}^{n}(-1)^{k+R}\,{}_{k-1}C_{R-1}S_k\end{aligned}$$

From these formulas, we can find the following:

$$\begin{aligned}(N_{\text{I}})_0&=S_1-{}_2C_1S_2+{}_3C_1S_3-{}_4C_1S_4+\cdots\\&=\sum_{k=1}^{n}(-1)^{k+1}\,{}_kC_1S_k\end{aligned}$$

$$\begin{aligned}(N_{\text{II}})_0&=S_2-{}_3C_2S_3+{}_4C_2S_4-{}_5C_2S_5+\cdots\\&=\sum_{k=1}^{n}(-1)^k\,{}_kC_2S_k\end{aligned}$$

$$\begin{aligned}(N_{\text{III}})_0&=S_3-{}_4C_3S_4+{}_5C_3S_5-{}_6C_3S_6+\cdots\\&=\sum_{k=3}^{n}(-1)^{k+1}\,{}_kC_3S_k\end{aligned}$$

$\cdots$

$$(N_R)_0=\sum_{k=R}^{n}(-1)^{k+R}\,{}_kC_RS_k$$

The proof of these formulas depends on the following relations:

$$\begin{aligned}S_1&=(N_{\text{I}})_0+{}_2C_1(N_{\text{II}})_0+{}_3C_1(N_{\text{III}})_0\\&\quad+{}_4C_1(N_{\text{IV}})_0+\cdots\\S_2&=(N_{\text{II}})_0+{}_3C_2(N_{\text{III}})_0+{}_4C_2(N_{\text{IV}})_0\\&\quad+{}_5C_2(N_{\text{V}})_0+\cdots\\S_3&=(N_{\text{III}})_0+{}_4C_3(N_{\text{IV}})_0+{}_5C_3(N_{\text{V}})_0\\&\quad+{}_6C_3(N_{\text{VI}})_0+\cdots\\S_4&=(N_{\text{IV}})_0+{}_5C_4(N_{\text{V}})_0+{}_6C_4(N_{\text{VI}})_0\\&\quad+{}_7C_4(N_{\text{VII}})_0+\cdots\end{aligned}$$

$\cdots$

From a consideration of the manner in which the positive characteristic frequencies are derived from the ultimate set frequencies, these relations can be seen to hold.

From the above, we can formulate the indicated linear combinations of the S_i and show that they reduce to the required values. As an example, let us take N_{II+}. The proof consists in substituting the S_i as given above into:

$$N_{II+}=S_2-2S_3+3S_4-4S_5+\cdots$$

and showing that the resultant expression reduces to

$$(N_{II})_0+(N_{III})_0+(N_{IV})_0+(N_V)_0+\cdots$$

which is the correct value of N_{II+}.

After substituting, we have the following expression:

$$\begin{aligned}N_{II+}=(N_{II})_0&+[{}_3C_2-{}_2C_1\,{}_3C_3](N_{III})_0\\&+[{}_4C_2-{}_2C_1\,{}_4C_3+{}_3C_1\,{}_4C_4](N_{IV})_0\\&+[{}_5C_2-{}_2C_1\,{}_5C_3+{}_3C_1\,{}_5C_4\\&\qquad-{}_4C_1\,{}_5C_5](N_V)_0+\cdots\end{aligned}$$

It can be shown that each of the coefficients is equal to unity by virtue of the following general combinatorial relation:

$${}_kC_r\,{}_rC_{k+1}-{}_{k+1}C_r\,{}_rC_{k+2}+{}_{k+2}C_r\,{}_rC_{k+3}+\cdots\pm{}_{r-1}C_k\,{}_rC_r=1$$

We may now return to the problems proposed originally in this section. For the first, we have

$$S_1=178,\quad S_2=40,\quad S_3=10.$$

(1) The number of students engaging in one or other of the given activities is

$$N_{1+}=S_1-S_2+S_3\cdots=178-40+10=148$$

(2) The number of students not participating in any activities is thus

$$762-148=614$$

(3) The number having exactly one activity is:

$$(N_1)_0=S_1-2S_2+3S_3=178-80+30=128$$

(4) The number having exactly two activities is:

$$(N_{II})_0=S_2-3S_3=40-30=10$$

In the second problem, the given data are $N_1=125$, $N_2=87$, $N_3=75$, $N_0=45$, and $N_{II+}=76$. Also, $N=235$. We wish to find $S_3=(N_{III})_0$. We use the following two relations:

$$N=N_0+S_1-S_2+S_3$$
$$N_{II+}=S_2-2S_3$$

Substituting known values:

$$\begin{aligned}235&=45+287-S_2+S_3\\76&=\qquad\quad S_2-2S_3\end{aligned}$$

Eliminating S_2 and solving for S_3, we find that $S_3=21$.

And now having set up the heavy artillery, we shall throw up a few clay mosquitoes which our readers may find some pleasure in demolishing.

Problems

1. In a race involving five horses, 1,324 of the 4,562 people attending the race abstained from betting. The total number of bets placed was 5,142 (one bet is considered to be one person wagering on one horse; if the same person bets on two horses, this is two bets in the meaning of this problem). No one bet on four or five horses in this race, while 2,142 laid money on just one horse.

   a) How many people bet on exactly two horses?

   b) How many on exactly three horses?

2. Check the following data for consistency by determining the ultimate class frequencies:

$$N_0=64,\quad N_1=85,\quad N_2=72,\quad N_3=48,$$
$$N_{12}=63,\quad N_{13}=37,\quad N_{23}=32,\quad N_{123}=21.$$

3. A door-to-door salesman sold four articles: (1) *A*, worth \$20; (2) *B*, worth \$75; (3) *C*, worth \$150; (4) *D*, worth \$200. His regular procedure was to try to sell them in sequence from the less valuable to the more valuable. After 1,000 calls he had the following data for these sets:

$$N_1=66,\quad N_2=45,\quad N_3=27,$$
$$N_4=22,\quad N_{12}=23,\quad N_{13}=11,$$
$$N_{14}=4,\quad N_{23}=10,\quad N_{24}=7,$$
$$N_{34}=2,\quad N_{123}=2.$$

a) How many people bought nothing from him?
b) How many bought exactly one item?
c) How many bought two items?
d) What per cent of those who purchased item A bought item B, but nothing more?

4. In a large family gathering, it was found that there were 10 more minors than adults; there were 12 boys, 6 married persons of the female sex, and 4 adult women who were not married. At dinner, the unmarried girls, eight in number, were placed at a special table. (Note: The terms "boys," and "girls," signify minors; it may also be pointed out that persons less than 21 can be married.) If there are more persons of the female sex than minors at the gathering, what is the largest number of persons in all that could be present?

5. A survey was made of a group of 983 families to ascertain which of them had (1) an automobile; (2) a television set; (3) an electric washer; (4) a radio. It was found that no family was completely without such equipment; that 481 families had at least two of these items; 345 had at least three; and 264 had all four.

a) How many pieces of equipment did all 983 families have?
b) How many families had (1) exactly one; (2) exactly two; (3) exactly three pieces of equipment?

6. Given the following data as determined from a tabulation for four sets, find the positive characteristic frequencies.

$$N_0=42, \quad (N_1)_0=28, \quad (N_2)_0=33,$$
$$(N_3)_0=42, \quad (N_4)_0=27, \quad (N_{12})_0=15,$$
$$(N_{13})_0=13, \quad (N_{14})_0=10, \quad (N_{23})_0=21,$$
$$(N_{24})_0=13, \quad (N_{34})_0=8, \quad (N_{123})_0=10,$$
$$(N_{124})_0=12, \quad (N_{134})_0=18, \quad (N_{234})_0=7,$$
$$(N_{1234})_0=8$$

Check the results for ΣN_i, ΣN_{ij}, ΣN_{ijk} and for N.

7. In a school having a population of 537, all taking the academic subjects of mathematics, English, social studies, and foreign language, a survey was made to determine which subjects the students liked. The statistics indicated a total of 172 who liked mathematics; 225 who liked English; 235, social studies; and 175, foreign language. A total of 57 expressed a liking for both mathematics and English, 63 for mathematics and social studies, 50 for mathematics and foreign language, 90 for English and social studies, 76 for English and foreign language, and 69 for social studies and foreign language. There were 26 who liked mathematics, English, and social studies; 23, mathematics, English, and foreign language; 21, mathematics, social studies, and foreign language; 37, English, social studies, and foreign language. There were 15 who said they liked all four subjects.

a) How many students liked none of these studies?
b) How many did not like mathematics?
c) What per cent of those who liked English did not like mathematics?
d) What per cent of those who liked mathematics did not like English?
e) How many liked mathematics but no other subject? English only? social studies only? foreign language only?
f) What per cent of those who liked mathematics and foreign language did not like English and social studies?
g) What per cent of the students liked two or more subjects?

8. In a sampling of 1,000 people, it was found that 215 took at least two of four leading national magazines, 35 took all four, while 125 took none. In all, 965 copies were involved. Check these data for consistency.

9. A ship sailing for Hawaii had aboard 517 passengers, of whom 87 were American women, 45 foreign boys, 378 adults, 132 American men, 140 foreign males, and 22 foreign girls. If there were 64 more Americans than foreigners aboard, determine the following:

a) The number of foreign children
b) The number of Americans
c) The number of foreign adults
d) The number of men
e) The number of women
f) The number of foreign men
g) The number of foreign women
h) The number of American children

10. Given sets A_1, A_2, A_3 represented by circles such that A_1 includes areas marked 2, 3, 5, 6; A_2, areas marked, 3, 4, 5, 7; and A_3, areas marked 5, 6, 7, 8; determine what areas correspond to the following sets:

a) A_{12}; b) A_{23}; c) A_{123}; d) $A_{\bar{3}}$; e) $A_{1\bar{3}}$; f) $A_{\bar{2}3}$

11. The students in a school of 527 enrollment made a survey to determine the feelings of students about the top five tunes. It was found that none of the students was without knowledge of such and that all liked at least one of the numbers. A total of 325 expressed their approval of two or more of the selections, while 26 liked at least four and 4 approved all five. In all, there were 842 expressions of approval from the entire group for one or other of the numbers. Such was the report of the committee. Do these figures stand the test of consistency?

Right triangle construction

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A helpful table for the teacher who wants to use problems about right triangles and still avoid the familiar 3, 4, 5 right triangle.

THE PERIODIC classroom tests given by algebra, geometry, and trigonometry teachers invariably contain drawings of triangles to be solved. Some students are confused if their solutions do not agree with the *shape* of the drawn triangle. To prevent constant usage of the hackneyed 3, 4, 5 right triangle and its multiples, a teacher can easily form Pythagorean triples[1] which can add endless varieties of right triangles to the test paper and have the advantage that *only natural numbers* are used as sides. These natural numbers can be taken as integral distances easily measured with a triangular rule or architect's scale.

In this development the following theorem was used.[2] If m and n are any natural numbers, $m>n$, without common factor, and not both odd, then the natural numbers form a primitive Pythagorean triple, according to the following formulas:

$$c^2=a^2+b^2 \tag{1}$$

where

$$a=m^2-n^2 \tag{2a}$$

$$b=2mn \tag{2b}$$

$$c=m^2+n^2 \tag{2c}$$

[1] For a previous illuminating article, see Ben Moshan, "Primitive Pythagorean Triples," THE MATHEMATICS TEACHER, November, 1959, pp. 541–545.

[2] G. M. Merriman, *To Discover Mathematics*, (New York: John Wiley and Sons, Inc., 1942), pp. 42–48.

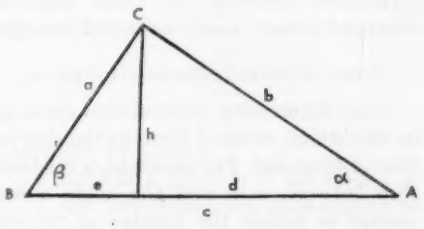

Figure 1

The table at the end of this article is a table of Pythagorean triples from which one can pick a likely triangle, measure and draw the hypotenuse c. To finish construction of the legs a and b, he can then draw arcs with B and A as centers, respectively, using a compass and the same scale used to measure c. Connecting the intersection of the arcs with points A and B forms a right triangle.

The table is oriented for those teachers who wish to emphasize a certain angle, 3°–89°, to the nearest degree. The $\angle\alpha$ represents the angle between side b and hypotenuse c in all cases except those in which the $\angle\alpha$ was interchanged with its complement $\angle\beta$, for the sake of condensing the table. All interchanged angles are in italics.

These triples represent the smallest natural numbers under 1,000 (for easy scale reading) which can be taken as the sides of right triangles whose interior angles are measured to the nearest degree.

An auxiliary table is included in the table, namely h, the altitude on the hypotenuse of the right triangle, and d, the distance of the foot of the altitude from the vertex of the $\angle\alpha$. Since $\angle\alpha$ is measured to the nearest degree, slight discrepancies are bound to creep in (e.g., h should be equal to d for $\angle 45°$), but for the purposes of these triangles these errors are not critical. This article is designed to help the classroom teacher in mathematics make rather accurate diagrams on classroom tests, and add his own spice of variety to these tests. The table may be used as a permanent reference by those teachers who need a ready, easily measured triangle.

Some interesting relationships

Many interesting relationships came up in this study, most of them in the derivation of formulas. For example, a relationship between m/n and the angle α was needed to reduce the number of calculations. This was the result.

If $b=2mn$ and $c=m^2+n^2$, then

$$\frac{c}{b}=\frac{m^2+n^2}{2mn}=\frac{1}{2}\left(\frac{m^2}{mn}+\frac{n^2}{mn}\right),$$

or

$$\frac{c}{b}=\frac{1}{2}\left(\frac{m}{n}+\frac{1}{\frac{m}{n}}\right).$$

Let $g=m/n$. Then

$$\frac{c}{b}=\frac{1}{2}\left(g+\frac{1}{g}\right)=\frac{g^2+1}{2g}.$$

But, referring to Figure 1, we see that

$$\frac{c}{b}=\frac{1}{\cos\alpha}=\sec\alpha=\frac{g^2+1}{2g}$$

or,

$$g^2+1=2g\sec\alpha$$

and

$$g^2-2g\sec\alpha+1=0.$$

This equation is now in the form $Ax^2+Bx+C=0$. We can use the quadratic formula

$$x=\frac{-B\pm\sqrt{B^2-4AC}}{2a}$$

where $x=g$, $A=+1$, $B=-2\sec\alpha$, and $C=+1$.

Substituting, we get

$$g=\frac{-(-2\sec\alpha)\pm\sqrt{(-2\sec\alpha)^2-4(+1)(+1)}}{2(+1)}$$

$$g=\frac{+2\sec\alpha\pm\sqrt{4\sec^2\alpha-4}}{2}$$

$$g=\frac{2\sec\alpha\pm\sqrt{4(\sec^2\alpha-1)}}{2}.$$

Since $\tan^2\alpha=\sec^2\alpha-1$,

$$m/n=\sec\alpha+\tan\alpha. \tag{3}$$

(Reject negative root.)

But,

$$\ln\frac{m}{n}=\ln(\sec\alpha+\tan\alpha)=\int_0^\alpha \sec t\,dt \tag{4a}$$

or,

$$\frac{m}{n}=\exp\int_0^\alpha \sec t\,dt. \tag{4b}$$

Equations 4a and 4b are interesting facts but have no bearing on finding a direct relationship between a single function of α and m/n.

Since

$$\sec\alpha+\tan\alpha=m/n,$$

then

$$\frac{1}{\cos\alpha}+\frac{\sin\alpha}{\cos\alpha}=\frac{m}{n}$$

or,

$$\frac{1+\sin\alpha}{\cos\alpha}=\frac{m}{n}.$$

This is a convenient function if we use $\alpha=90°-\beta$. This leads to

$$\frac{1+\sin(90°-\beta)}{\cos(90°-\beta)}=\frac{m}{n}$$

or,

$$\frac{1+\cos\beta}{\sin\beta}=\frac{m}{n}.$$

Since

$$\frac{1+\cos\beta}{\sin\beta}=\cot\frac{1}{2}\beta$$

then

$$\cot\frac{1}{2}\beta=\frac{m}{n},$$

$$\frac{1}{2}\beta=\text{Arc cot}\frac{m}{n},\text{ and}$$

$$\beta=2\text{ Arc cot }m/n. \qquad (5)$$

Equation (5) became the formula used to find the angles in the table.

m	n	m/n
2	1	2.00
3	2	1.50
4	1	4.00
4	3	1.33
5	2	2.50
5	4	1.25
6	1	6.00
6	3	2.00
6	5	1.20
7	2	3.50
7	4	1.75
7	6	1.67
8	1	8.00

Values of m and n were placed in a table such as the one above in accordance with the basic theory (either m or n must be odd, m greater than n, no common factors). From a standard trigonometric table, one must find the angle whose cotangent is equal to the ratio m/n, and multiply the result by two. This result is the angle β. The complement of angle β is the angle α.

Two interesting relationships developed while finding the altitude h and the distance d. Referring to Figure 1, it may be seen that b is a mean proportional between c and d, or

$$\frac{d}{b}=\frac{b}{c},$$

$$dc=b^2, \qquad (6)$$

$$d=\frac{b^2}{c}.$$

Therefore, $d=b\cos\alpha$.

To find h as a function of the sides a, b, and c, find two expressions with h and a as mean proportionals.

$$\frac{e}{n}=\frac{h}{d} \qquad \frac{e}{a}=\frac{a}{c}$$

$$h^2=ed \qquad (7) \qquad ec=a^2 \qquad (8).$$

$$e=\frac{a^2}{c}$$

Substituting formulas (6) and (8) into equation (7), we obtain

$$h^2=\frac{a^2}{c}\cdot\frac{b^2}{c}=\frac{a^2b^2}{c^2}$$

or

$$h=\frac{ab}{c}=a\cos\alpha.$$

A check on the calculations made comes when we again consider equation (3):

$$m/n=\sec\alpha+\tan\alpha$$

$$\frac{m}{n}=\frac{c}{b}+\frac{a}{b}=\frac{c+a}{b}.$$

TABLE OF PYTHAGOREAN TRIPLES

α in degrees	m	n	m/n	(a) m^2-n^2	(b) $2mn$	(c) m^2+n^2	b/c $\cos\alpha$	$\lambda=ab/c$ $a\cos\alpha$	$d=b^2/c$ $b\cos\alpha$	β in degrees
3	17	16	1.0625	33	544	545	.99817	32.9	543.0	87
4	14	13	1.0769	27	364	365	.99726	26.9	363.0	86
5	11	10	1.1000	21	220	221	.99548	20.9	219.0	85
6	10	9	1.1111	19	180	181	.99448	18.9	179.0	84
7	9	8	1.1250	17	144	145	.99310	16.9	143.0	83
8	8	7	1.1428	15	112	113	.99115	14.9	111.0	82
9	7	6	1.1667	13	84	85	.98824	12.8	83.00	81
10	6	5	1.2000	11	60	61	.98361	10.8	59.0	80
11	10	1	10.0000	99	20	101	*.19802*	19.6	4.0	*79*
12	16	13	1.2308	87	416	425	.97882	85.2	407.2	78
13	5	4	1.2500	9	40	41	.97561	8.8	39.0	77
14	8	1	8.0000	63	16	65	*.24615*	15.5	3.9	*76*
15	13	10	1.3000	69	260	269	.96654	66.7	251.3	75
16	4	3	1.3333	7	24	25	.96000	6.7	23.0	74
17	13	2	6.5000	165	52	173	*.30058*	49.6	15.6	*73*
18	11	8	1.3750	57	176	185	.95135	54.2	167.4	72
19	6	1	6.0000	35	12	37	*.32432*	11.4	3.9	*71*
20	10	7	1.4286	51	140	149	.93960	47.9	131.5	70
21	11	2	5.5000	117	44	125	*.35200*	41.2	15.5	*69*
22	21	4	5.2500	425	168	457	*.36761*	156.2	61.8	*68*
23	3	2	1.5000	5	12	13	.92308	4.6	11.1	67
24	14	3	4.6667	187	84	205	*.40976*	76.6	34.4	*66*
25	9	2	4.5000	77	36	85	*.42353*	32.6	15.2	*65*
26	8	5	1.6000	39	80	89	.89888	35.1	71.9	64
27	13	8	1.6250	105	208	233	.89270	93.7	185.7	63
28	4	1	4.0000	15	8	17	*.47059*	7.1	3.8	*62*
29	12	7	1.7143	95	168	193	.87047	82.7	146.2	61
30	15	4	3.7500	209	120	241	*.49793*	104.1	59.8	*60*
31	7	4	1.7500	33	56	65	.86154	28.4	48.2	59
32	7	2	3.5000	45	28	53	*.52830*	23.8	14.8	*58*
33	10	3	3.3333	91	60	109	*.55046*	50.1	33.0	*57*
34	13	4	3.2500	153	104	185	*.56216*	86.0	58.5	*56*
35	16	5	3.2000	231	160	281	*.56940*	131.5	91.1	*55*
36	28	9	3.1111	703	504	865	*.58266*	409.6	293.7	*54*
37	2	1	2.0000	3	4	5	.80000	2.4	3.2	53
38	23	8	2.8750	465	368	593	*.62057*	288.6	228.4	*52*
39	14	5	2.8000	171	140	221	*.63348*	108.3	88.7	*51*
40	11	4	2.7500	105	88	137	*.64234*	67.4	56.5	*50*
41	8	3	2.6667	55	48	73	*.65753*	36.2	31.6	*49*
42	9	4	2.2500	65	72	97	.74227	48.2	53.4	48
43	16	7	2.2857	207	224	305	.73443	152.0	164.5	47
44	5	2	2.5000	21	20	29	*.68966*	14.5	13.8	*46*
45	12	5	2.4000	119	120	169	.71006	84.5	85.2	45

*Remarks on the teaching of calculus in the secondary school**

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What mathematics should be offered to students in the twelfth grade?

THE MATHEMATICAL professions are now devoting a great deal of attention to the curriculum in the grades: there are the programs of the Commission on Mathematics of the College Entrance Board, the School Mathematics Study Group (SMSG), and the University of Illinois Examination Committee on School Mathematics, to name a few. It is a good time to look at our problems in perspective and ask what we hope to accomplish in the lower curriculum for the purpose of directing the evolution of that curriculum.

In the grades we should try to build the broadest possible base of concrete experience for the student's later development in technology, science, philosophy, and mathematics itself. The student's development is not to be aimed at any specialty; he is to seek a foundation of learning which will enable him to move in any direction according to his abilities and tastes. The particular problem of today's discussion is so deeply related to this general conception of the curriculum that we may well analyze it to set up more specific guides for thought.

Before we seek to direct the evolution of the present curriculum, we must ask in what respects it fails. The criticism of mathematics as it is now commonly taught falls primarily under two headings, which I shall call "liberal" and "technical." The liberal objections are that the teaching of mathematics is too dogmatic; too much devoted to rule-of-thumb and memorization, too little devoted to intuition, argument, reason, and deduction. The technical objections are primarily that the student is not prepared for an early entrance into the sciences, but also that such preparation as he has is poorly digested and not seen in relation to the problems to which it applies. The experienced teacher knows well that these objections are linked and that he will do better in trying to meet both classes of criticism simultaneously than one alone. What better way of imparting the intuitiveness and rationality of mathematical principles than by the ample development of interesting applications to the sciences and to other mathematics? What better way of teaching the values of mathematical formalism and deduction than by applying these methods to a problem for which the solution could hardly be foreseen by a naïve intelligence?

Turning to specific possibilities, it is easy to see how the teaching of arithmetic in the elementary grades could be informed and unified, most notably by the gradual introduction of the literal symbolism of algebra, perhaps beginning in the fourth grade, proceeding in gentle stages so that at the end of the sixth grade

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the idea of arithmetical operations with literal symbols is as familiar as the process of counting. The applicability and interest of primary mathematics would be greatly enhanced thereby without prohibitive effort.

In looking to the future secondary curriculum, let us first take note of the fact that we may hope for a certain amount of extra freedom. The mathematics taught in grades 7 and 8 is being examined. The endless calculation of percentages, so dull to teacher and student alike, may well be replaced by a course containing some number theory, intuitive algebra, and intuitive geometry. The SMSG texts for the junior high school years represent a welcome step in that direction. If this change finds general acceptance (and I think we ought to push hard for it), then by careful planning we may have the gift of an extra high school year to devote to instruction beyond the present level. Let me assume that the evolution of the curriculum is pushed in that direction and treat the gift of that extra year as though it will become fact.

To formulate our ideals in terms of a concrete objective for a universal secondary curriculum, we must at least consider the various demands of society outside the profession. In the physical sciences and in engineering, there is widespread sentiment for a curriculum which prepares for the calculus in the freshman college year, or better, a curriculum which terminates with calculus. Even in the natural sciences and the social studies the preferred background is the calculus. In business management and economics, for example, the calculus is valued not least for the fact that it provides the language for the formulation of a large class of problems. There is no need to amplify on the role of the calculus in the training of every mathematician. The traditional curriculum up through analytic geometry is clearly adaptable to this need, although there surely will be certain modifications in detail. The preparation for the calculus could be accomplished by the end of the third year of senior high school, pointing to a quite definite subject matter which we may hope by degrees to make into the standard curriculum for grades 7 through 11. All of this leads to the question of what we shall offer to those students who are energetic enough to elect another year of mathematics in grade 12.

It is of course, calculus, and a full year of it, that I propose as the proper matter for grade 12. There is no other subject which so suitably rounds out the elementary curriculum. The concepts, methods and skills developed in the earlier subjects find application in an interesting and meaningful way so that the student's mastery of his earlier learning is strengthened. In its manipulative and problem-solving aspects, too, the calculus is very close to the spirit of secondary school mathematics. The student is brought to the threshold of higher mathematics, physics, engineering, and a wealth of other applications. He has only to open the door and walk in.

Why not something else? Why not probability and statistics, or an introduction to modern algebra, or a semester on elementary functions followed by a semester of linear algebra? Stripped of detail, my answer is simply that the calculus provides that broadest possible base of experience for the student's later development which we seek. Hardly a branch of science, technology, and even social science fails to benefit from the application of the concepts of the calculus, often in matters of fundamental significance. The calculus plays a profound role in the intellectual and practical affairs of men, either directly, or implicitly as part of the necessary background in the development of other knowledge. There is no need to emphasize the central role the concepts of the calculus play in all the later ramifications of higher mathematics.

So far, I have only presented an ideal. Now for the practicalities! What kind of calculus course is it to be? Where are the teachers to be found? Where are the texts?

How will the colleges treat the new crop of students? In fact, can their high school preparation in calculus be considered adequate for advanced standing? Isn't it really impractical to propose such a drastic reorganization of high school mathematics?

Let me emphasize that no drastic immediate reorganization is proposed. The curriculum we are discussing is to be the culmination of a period of planned evolution. There are high schools teaching calculus now. There will be more. It is not enough to leave the development to chance, however. We should seek to have such a curriculum in the larger schools in five years, and to have it generally in a decade. It is important to set fixed goals like these and plan in these terms; otherwise the objectives will be lost through inaction or indirection. Once we accept an evolutionary development the rest does not seem so difficult.

In the first place, the present education of our teachers makes it more likely that they have had the calculus than any of the alternatives commonly proposed; it does not seem unreasonable to presuppose that the present ranks will be adequate to supply the immediate need while new recruits are trained. It ought to be added that the teachers of high school mathematics feel more qualified to teach the calculus or topics from the calculus than any of the proposed alternatives. Texts exist—not many good ones, but if the program meets acceptance, new ones will be written. I should hope that the introduction of the calculus will in time provide a general elevation of the curriculum so that the present mountainous accumulation of meretricious texts would be supplanted. I should expect that the training of teachers of mathematics would also undergo general improvement. Again, these are matters which cannot be left to chance; we shall have to work hard to attain those ends.

The course itself I envision as being in the best secondary school tradition. It would be essentially an intuitive calculus, foregoing a precise epsilontic statement of the concept of limit, for example. However, the student should be able to obtain clear insights into certain proofs; and even though he may not be able to complete the punctuation of a proof, he can be made aware that it is possible and desirable to do so. It is more important that he have a keen feeling for the basic concepts and their applicability and be able to formulate problems in terms of these concepts. A heightened appreciation of the necessity for precision of statement and logical reasoning can only be obtained for most people after the amassing of considerable detailed experience. Rigor may reasonably be left to the colleges. In England, it is done that way.

The colleges may be reluctant to immediately admit the high school students with this four year training to their more advanced courses. The question can easily be settled by means of a qualifying examination or personal interview. A student who is admitted with advanced standing should expect and receive a calculus course in which the need for a more careful development is demonstrated and met.

Incidentally, this program should not be conceived as being only for the college-bound. Each student should be led as far along the road as is reasonably possible. Each stage mastered will multiply his opportunities in this highly technical culture. Good advisory service is certainly needed here.

I must say, I look forward with relish to the prospect of having students of the new era in my classes.

To conclude I should like to borrow some words from E. H. Moore dating back to the turn of the century: "Will not the twentieth century find it possible to give to young students during their impressionable years in thoroughly concrete and captivating form the wonderful new notions of the seventeenth century?" We have forty years to go. Let's try to make it in ten.

An interesting problem involving indeterminate equations

JESSE C. LEWIS, *Jackson State College, Jackson, Mississippi.*

In this simple but intriguing example the domain of the variables is the set of positive integers and zero.

GIVEN ONE equation involving two variables, x and y, it is clear that by ascribing any value we please to x, we can obtain a corresponding value for y; thus the equation has an infinite number of solutions. However, for many problems it is often clear from the nature of the problem that the solutions must be positive integers; with this restriction, as we shall see later, the number of solutions is limited.

If the number of variables is greater than the number of independent equations, there will be an unlimited number of solutions, and the equations are said to be indeterminate. The problem discussed in this article will involve only the simplest kind of indeterminate equations and will allow us to confine our attention to zero and positive integral values of the variables. It will be seen that this restriction enables us to express the solutions in a very simple form.

Suppose the following problem were proposed for solution:

Find all the ways a dollar can be changed into 21 pieces (coins).

Let P be the number of pennies, N the number of nickels, D the number of dimes, Q the number of quarters, and H the number of half dollars.

We can derive two independent equations from the problem. So let us begin by changing the dollar into 21 pieces using only pennies, nickles, and dimes. Thus we have

$$P+5N+10D=100, \qquad (1)$$

$$P+N+D=21. \qquad (2)$$

Subtracting (2) from (1), we have

$$4N+9D=79.$$

Divide throughout by 4, the smallest coefficient; thus

$$N+2D+\frac{D-3}{4}=19. \qquad (3)$$

Since N and D are to be integers, we must have

$$\frac{D-3}{4}=\text{integer};$$

let us call this integer x, and write

$$\frac{D-3}{4}=x;$$

therefore

$$D-3=4x,$$

or

$$D=4x+3. \qquad (4)$$

Substituting this value of D in (3),

$$N+8x+6+x=19;$$

that is,

$$N=13-9x. \qquad (5)$$

If in these results we give to x any integral value, we obtain corresponding integral values of N and D; using these values in (1) or (2), we get an integral value for P. But if x is greater than 1, we see from (5) that N is negative; and if x is a negative integer, D is negative. Thus the only

positive integral values of N and D are obtained by putting $x=0,\ 1$.

The complete solution may be exhibited as follows:

x	P	N	D
0	5	13	3
1	10	4	7

To find the solutions involving quarters, dimes, nickels, and pennies, we proceed as follows:

Given one quarter, the problem reduces changing 75 cents into 20 pieces. Thus we have

$$P+5N+10D=75, \qquad (1)$$

$$P+N+D=20. \qquad (2)$$

Subtracting (2) from (1), we have

$$4N+9D=55.$$

Dividing throughout by 4, we have

$$N+2D+\frac{D-3}{4}=13. \qquad (3)$$

Since N and D are to be integers, we must have

$$\frac{D-3}{4}=\text{integer, say } x;$$

therefore

$$D-3=4x,$$

or

$$D=4x+3.$$

Substituting this value of D in (3), we have

$$N=7-9x.$$

The only positive integral value of N and D is obtained by setting $x=0$. Thus we exhibited the complete solution:

x	P	N	D	Q
0	10	7	3	1

Given two quarters, the equations are:

$$P+5N+10D=50, \qquad (1)$$

$$P+N+D=19. \qquad (2)$$

Subtracting (2) from (1), we have

$$4N+9D=31.$$

Dividing by 4, we have

$$N+2D+\frac{D-3}{4}=7. \qquad (3)$$

As before, we must have

$$\frac{D-3}{4}=\text{integer, } x;$$

therefore

$$D-3=4x,$$

or

$$D=4x+3.$$

Substituting this value of D in (3), we have

$$N=1-9x.$$

The only positive integral value of N and D is obtained by setting $x=0$. Therefore the complete solution is

x	P	N	D	Q
0	15	1	3	2

Given one half dollar and one quarter, the two independent equations are:

$$P+5N+10D=25, \qquad (1)$$

$$P+N+D=19. \qquad (2)$$

Subtracting (2) from (1) and dividing by 4, we have

$$N+2D+\frac{D-3}{4}=1.$$

Since

$$N,\ 2D,\ \text{and}\ \frac{D-3}{4},$$

are to be positive integers, their sum cannot be equal to one. Therefore we conclude that there is no solution involving both a quarter and a half dollar.

To obtain the fifth and final solution, we assume one half dollar and write:

$$P+5N+10D=50, \quad (1)$$

$$P+N+D=20. \quad (2)$$

Subtracting (2) from (1) and dividing by 4, we have

$$N+2D+\frac{D-2}{4}=7. \quad (3)$$

Again we must have

$$\frac{D-2}{4}=\text{integer}, x;$$

therefore

$$D=4x+2.$$

Substituting this value of D in (3), we have

$$N=3-9x.$$

Setting $x=0$ gives the only solution, exhibited as follows:

x	P	N	D	H
0	15	3	2	1

Have you read?

ORE, OYSTEIN, "Pascal and the Invention of Probability Theory," *The American Mathematical Monthly*, May 1960, pp. 409–419.

As you introduce the topic of probability to your students, have them read this very readable article. It brings out the interesting personal relationship that existed between De Mire, the prominent humanitarian of the court of Louis XIV, and Pascal. It relates how they traveled together and discussed mathematics. You will be interested to note how De Mire takes the credit for starting Pascal on the road to fame in the field of probability. The article corrects previous concepts as to why De Mire wanted Pascal to help him establish the number of throws of dice needed for an advantageous proposition. Pascal's letter to Fermat, as quoted in this article, also raises some interesting questions.

The second problem De Mire proposed is the division problem, that is, division of prize money if a program is closed before completion.

This is an article you shouldn't miss because it gives a brief but lucid account of the environment in which a whole new branch of mathematics developed.—PHILIP PEAK, *Indiana University, Bloomington, Indiana.*

TELLER, EDWARD, "We're Losing by Inches, by Feet, by Yards," *This Week Magazine*, May 15, 1960, pp. 6–7, 34–36.

Measurement involves the lives of each of us; therefore, your students will be interested in this article which is a plea for the metric system. Is the metric system Russia's biggest weapon? Does the metric system enable engineers and scientists to speak the same language? Can students in Russian schools learn more because the system is more efficient? Is the American schoolboy prevented from seeing the relation between length, area and volume because of our nonsystematic system of measure? Will we lose the economic battle because of measurement? These and many other questions are raised and discussed. You will be interested in the pictures and historical material as well as the prediction of how long it will take to make the change to the metric system. Read the article and draw your own conclusion.—PHILIP PEAK, *Indiana University, Bloomington, Indiana.*

VOGELI, BRUCE R. "Recent Revisions in Soviet Mathematics Education," *School Science and Mathematics*, June 1960, pp. 425–438.

You have heard and read all types of reports as to just what is being taught in the mathematics programs of Russia. This article is short, to the point, and yet reasonably complete. The impact of the 1958 change is considered, major topics in mathematics are given for each type of school, and the difference between the old 10-year school and the new 11-year school is pointed out.

You will note that the total time spent on mathematics is over 2,000 hours or 17% of the student's time, that trigonometry is subsumed in other topics in the new program, that approximate computation is included, and that topics in solid geometry have been placed in plane geometry but not integrated with it. In general, pupils begin the study of both algebra and geometry simultaneously in grade 7. The revisions taking place reflect emphasis on polytechnic work and on concrete applications.

To bring yourself up to date on mathematics in Russia, you should read this article and the one preceding it in the April issue of *School Science and Mathematics*, "Elementary Mathematics Program in Russia."—PHILIP PEAK, *Indiana University, Bloomington, Indiana.*

Teaching seventh grade mathematics by television

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How does instruction in mathematics by television compare with mathematics instruction in the conventional classroom?

Introduction

How does instruction in mathematics by television compare with mathematics instruction given in the conventional classroom? Can television be used as a technique of in-service growth? Is there a differential in the effectiveness of television instruction upon varying levels of pupil ability? This study represents an attempt to pursue these problems within an experimental framework.

This experiment represents the first use of instructional television in the teaching of mathematics in the Cincinnati public schools. Teaching mathematics differs rather significantly from teaching many other subjects, in that successive learnings depend to a large extent upon past learnings. The area of science, in contrast, may be taught within several rather independent units. If a pupil fails to understand the content of one unit, he may to a large extent start afresh in another unit of science. The importance of mastering previous content is therefore emphasized in the teaching of mathematics.

Also, in relation to teaching science by television, teaching mathematics by television is less attractive in that only a few of the distinguishing characteristics of instructional television can be utilized. For example, the range of visuals that may be used in science to stimulate and to aid in the learning process is greater than it is in mathematics. The use of television in teaching mathematics, however, does permit the use of an outstanding teacher.*

An important aspect of this experiment is its attempt to determine whether a significant amount of in-service teacher growth has occurred as a result of a classroom teacher viewing the televised lessons. Due to the difficulty of measuring teacher growth directly, this measurement was inferred by the achievement of the pupils taught by the teacher. Admittedly, there are many limitations in this approach; however, its attractiveness lies in its objectivity and its amenability to measurement.

Description

In this experiment, a full year of instruction by television was offered. All telecasts were live presentations. Lessons were presented Monday, Wednesday, and Friday of each week for a period of 20 minutes. Each telecast was followed by 30 minutes of follow-up and discussion. On a weekly basis, therefore, 60 minutes of television instruction was presented in addition to 190 minutes of regular classroom instruction. Thus, 24 per cent of the total instructional time was devoted to television instruction.

* The television teacher in this experiment was Miss Marlene Beigel. She was capably guided in her efforts by Miss Mildred Keiffer and Mrs. Sarah Greenholz, supervisors of mathematics for the Cincinnati public schools.

The full year's program of instruction covered the following units:

Unit	Number of Lessons
Whole numbers	5
Common fractions	20
Decimal fractions	15
Per cent	21
Number systems	7
Measurement	20
Informal geometry	17
Graphs	13
Social arithmetic	17
Review	6
Total	141

Design of the Experiment

There are fifteen schools in the Cincinnati system where seventh grade mathematics is taught. Nine schools representative of the total were selected to participate in the experiment. Since classes are sectioned on the basis of pupil ability, principals were asked to select three classes adjacent to one another on a ranked list of the various class sections and representing an average of their school population.

A description of the three classes within each of the nine schools is as follows:

Class 1 (E_1). These classes are experimental classes represented by pupils taught partially by television along with a classroom teacher.

Class 2 (C_1). These classes are control classes represented by pupils who did not view television but who were taught by the same teacher serving E_1.

Class 3 (C_2). These classes are control classes represented by pupils who did not view television and who were taught by some teacher other than the E_1, C_1 teacher.

The three classes described above for each school were assigned randomly to one of the three treatment classifications, i.e., E_1, C_1, or C_2. After all administrative decisions regarding homogeneous groupings had been made, the E_1, C_1 teacher and the C_2 teacher were matched on the basis of teaching ability by the supervisors of mathematics. The same instructional content was followed by both experimental and control groups.

After all classes in the experiment were assigned, the mean IQ of all three classes within each of the nine schools was computed. The three schools with the highest mean IQ's were pooled and arbitrarily classified as above average in ability. The three schools with the next highest mean IQ's were pooled and arbitrarily classified as average in ability. The remaining three schools were arbitrarily designated as below average in ability.

The Sequential Test of Educational Progress, Mathematics, Level 3A, was used as the final criterion measure of achievement. Mental ability as reflected by intelligence test scores served as a basis for the use of the covariance technique. The intelligence test used was the Lorge-Thorndike Intelligence Test, Verbal, Level 3A. All tests were administered by trained examiners.

Methodology

The methods of covariance analysis were applied to the data in order to adjust for initial levels of mental ability. It should be made clear at this point, however, that the covariable of mental ability was studied not only for purposes of making adjustments in the final criterion means, but also in order to study the relationship between mental ability and the effectiveness of the methods of instruction. As indicated earlier, the three schools representing the highest mean ability level were pooled and arbitrarily classified as above average. A similar process was used for the average and below-average ability classifications. Homogeneity of variance and regression was established in order to insure the legitimacy of pooling. Subsequently the data were handled through a two-way classification of methods and ability levels. Since the number of pupils constituting each of the nine treatment classifications differed somewhat, the mean square for means was multiplied by the harmonic mean of the numbers in the treatment classifications so comparison with the within-groups error mean square

TABLE 1

AVERAGE IQ SCORES DERIVED FROM THE LORGE-THORNDIKE VERBAL INTELLIGENCE TEST FOR THE EXPERIMENTAL AND CONTROL CLASSES IN THE SEVENTH GRADE MATHEMATICS TELEVISION EXPERIMENT

Ability Level	*Methods Averages*							*Total Group Average*
	N	E_1*	*N*	C_1*	*N*	C_2*	*N*	
Above average	84	107.17	97	112.03	83	113.08	264	110.81
Average	93	108.65	84	103.54	84	96.23	261	103.00
Below average	81	91.48	71	94.56	67	93.88	219	93.21
Total	258	102.78	252	104.28	234	101.53	744	102.89

* For definition of these symbols, see page 544.

could be made. Finally, tests for homogeneous regression and variance were made, followed by a test for interaction between methods and ability levels.

RESULTS

The mean IQ's for the nine treatment classifications are shown in Table 1.

Inspection of the IQ means in Table 1 indicates the basis for the three ability level classifications; the mean IQ's for the above average, average, and below average ability levels are 110.81, 103.00, and 93.21, respectively. It is a chance variation in sampling that the above average E_1 group has a lower mean IQ than the average E_1 group. While the IQ means differs somewhat within levels and within the methods classifications, there are no a priori reasons for believing that a bias due to uncontrolled variables does exist. This does not preclude the possibility, however, that a bias could be present.

The average final criterion scores made by each of the nine classifications of pupils on the STEP, Mathematics, Level 3A, are shown in Table 2.

The first step in the analysis of these data was to compute the regression coefficients for each of the nine treatment classifications. These regression coefficients ranged from .1811 to .6322 with 7 out of these 9 coefficients being between .22 and .41. The highest regression coefficient occurred within the above average C_2 classification. The test for the hypothesis of homogeneous regression produced an F ratio of 1.91, d.f. 8, 727. The hypothesis of homogeneous regression was, therefore, accepted as tenable.

The hypothesis of no significant interaction effects was tested and subsequently

TABLE 2

AVERAGE SCORES DERIVED FROM THE SEQUENTIAL TEST OF EDUCATIONAL PROGRESS FOR MATHEMATICS, FORM 3A, FOR THE EXPERIMENTAL AND CONTROL CLASSES IN THE SEVENTH GRADE MATHEMATICS TELEVISION EXPERIMENT

Ability Level	*Methods Averages*							*Total Group Average*
	N	E_1	*N*	C_1	*N*	C_2	*N*	
Above average	84	67.77	97	71.58	83	72.24	264	70.58
Average	93	66.16	84	61.02	84	57.80	261	61.81
Below average	81	55.36	71	55.37	67	54.91	219	55.22
Total	258	63.29	252	63.49	234	62.09	744	62.98

TABLE 3

ADJUSTED MEANS, MEAN DIFFERENCES, AND SIGNIFICANCE OF MEAN DIFFERENCES AMONG EXPERIMENTAL AND CONTROL CLASSES IN THE SEVENTH GRADE MATHEMATICS TELEVISION EXPERIMENT

Ability Level	*Adjusted Methods Averages*			*Mean Differences*		
	E_1	C_1	C_2	E_1-C_1	E_1-C_2	C_1-C_2
Above average	65.63	68.50	68.80	−2.87*	−3.17*	−.30
Average	64.22	60.80	60.05	+3.42*	+4.17*	+.75
Below average	59.21	58.18	57.95	+1.03	+1.26	+.23
Total	63.10	63.02	62.52	+ .08	+ .58	+.50

* Difference significant at 5 per cent level.

rejected at the 5 per cent level of significance. The source of this significant interaction may be seen clearly in Table 3 which presents the adjusted means for each of the nine treatment classifications using the common within-groups regression coefficient.

The adjusted means shown in Table 3 were adjusted at the mean IQ of all groups, i.e., IQ 102.89. All the comparisons shown in Table 3, therefore, are made holding mental ability constant. Inspection of the mean difference columns points up clearly that the E_1 vs. C_1 and the E_1 vs. C_2 comparisons represent the sources of the significant interaction. The differences between the C_1 and C_2 comparisons are not only similar but lack significance as far as over-all difference is concerned. The latter comparisons indicate a lack of significant in-service teacher growth as reflected by pupil achievement.

In determining the significance of difference in mean differences, the critical difference was calculated at the 5 per cent level of significance. This critical difference was 2.77. The critical difference of 2.77 indicates simply that any difference below this figure may be due to sampling error in more than 5 times out of 100, while any difference equal to or exceeding 2.77 could be due to chance in 5 or less than 5 times out of 100. Four differences in Table 3 exceed the critical difference. All of these differences involve E_1 in the comparison. Furthermore, all differences are seen to occur in the above average and average ability level classifications.

These data indicate that both control methods of instruction were significantly more effective than the television method of instruction for classes of above average ability. Conversely, the television method of instruction is seen to be significantly more effective than the control methods of instruction for classes of average ability. At below average ability level, differences lack significance at the 5 per cent level.

In order to obtain a clearer picture of the data shown in Table 3, Figure 1 is presented. Figure 1 graphically represents the mean criterion score at each of the three levels adjusted at the mean IQ corresponding to each of the three ability levels, respectively. The means shown in Figure 1 differ from those shown in Table 3 simply because comparisons are made at each of three different IQ levels. The relationships in terms of differences, however, are exactly the same.

The proximity of the C_1, C_2 regression lines reflects the lack of significant differences among these groups, whereas the distance between the E_1 and C_1, C_2 regression lines at the mean IQ of the average and above average classifications reflects the source of the significant interaction. The proximity of the three methods means at the mean IQ of the below average classification reflects the lack of significant differences among methods means at the below-average ability level.

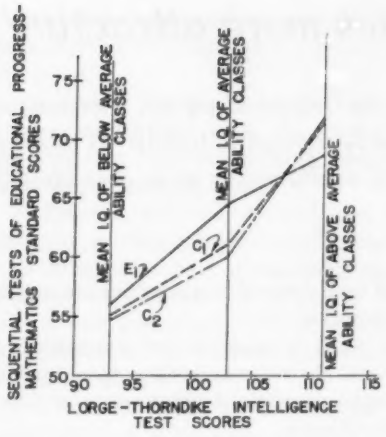


Figure 1

Criterion Test Means Adjusted in Relation to Mean Intelligence Test Scores for Below Average, Average, and Above Average Ability Levels for the Television and Non-Television Classes in the Seventh Grade Mathematics Experiment

Discussion and conclusions

The level of instruction by television in this experiment obviously was not directed toward the pupil of above average ability. It will be noted that at the above average level, the differences favored the C_1 group by 2.87 points and the C_2 group by 3.17 points. Television instruction would appear, however, to have been directed toward the pupil of average ability, as indicated by the difference of 3.42 for the E_1 vs. C_1 comparison and the difference of 4.17 for the E_1 vs. C_2 comparison, both differences favoring the television method.

Since television instruction was directed toward and benefited most the average ability pupil and appeared to be too slow for the above average ability pupil, one may question why the results did not indicate that the television method was not too fast for the below average ability pupil. One possible explanation for this asymmetry is that the above average ability pupil may be more sensitive to varied methods of instruction. The lack of significant difference between the C_1 and C_2 comparison at each of the three ability levels should not be interpreted as necessarily meaning a total absence of classroom teacher growth as a result of viewing the series of telecasts. It is possible, for example, that better teaching resulted without being reflected through pupil achievement on the criterion test. Perhaps it is necessary for teachers to be exposed to television for more than one year before a significant amount of in-service teacher growth is reflected in higher pupil achievement.

The close proximity of the C_1, C_2 regression lines helped to establish reliability when comparisons were made with the experimental television groups.

The conclusions of these experiments may be described as follows:

1. Significant interaction effects were noted between levels of pupil ability and methods of instruction.
2. At the high ability level, performance was significantly in favor of the conventional classroom method of instruction, while at the average ability level, performance was significantly in favor of the television method of instruction. At the below average ability level, no significant difference between methods of instruction was found.
3. There was no significant difference at any of the three ability levels among pupils taught by teachers who did and did not view television. Using pupil achievement as the criterion, therefore, it appears that there was no significant in-service teacher growth noted as a result of teachers viewing the televised lessons.
4. The absolute differences between methods of instruction are greater from the average to the above average ability classifications than they are from the below average to average ability classifications, indicating asymmetry in terms of the relation between the effectiveness of methods of instruction and levels of pupil ability.

How to make mathematics more attractive[1]

H. M. DADOURIAN, *Trinity College, Hartford, Connecticut.*

Some suggestions for the teacher who is attempting to help his students find mathematics as interesting as he does himself.

THE REACTION of the student to the study of mathematics depends on the student himself and the totality of his environment, past and present. However, in this discussion we are interested only in that part of the environment created by the presentation of the subject in textbooks and in the classroom.

After many years of classroom experience and observation, I ascribe the troubles some students have with mathematics to four sources:

1. The average student's notion that the study of mathematics requires a special kind of talent, and that mathematics, beyond the elements of arithmetic, is of little or no value to him unless he is planning to become a scientist or an engineer. These two notions produce fear of the subject and apathy toward it.

2. Failure to associate mathematics with experience and common sense, which leads to mystification and befuddlement.

3. Lack of knowledge of how to study effectively, which results in waste of time and effort, and consequent discouragement.

4. Disorderly ways of attempting to solve problems, which expose the student to blunders and errors.

The first of these sources of trouble is created mainly by the general public, including teachers in fields other than mathematics, engineering and the natural sciences. For the other sources, teachers and textbooks of mathematics are largely responsible.

I want to make a few suggestions of general character and follow them up with specific suggestions which relate to different subjects.

1. Approach the abstract through the concrete, with an appeal to the student's experience and common sense.

2. Instead of introducing a new concept by a formal definition, use the definition to crystallize and epitomize the explanation of the concept.

3. Avoid topics which do not fall in the main line of progress of the subject. Also, avoid topics which can be treated more effectively in a higher subject or which are untimely. Introducing the symbolism of symbolic logic into a first course in algebra constitutes a good example of bad timing.

4. Do not underestimate or overestimate the student's capacity.

5. Since problem solving forms the major part of the study of school and college mathematics, the following suggestions should be helpful:

a) Do not require the student to solve a great many problems of the same type. While studying a mathematical topic or subject, the student has to make use of most of what he has learned before. This fact makes understanding of principles and methods of greater importance than the development of skills.

b) Solve illustrative examples in such a way that the analysis forms a straight chain of reasoning, not a tangled mass of links.

c) Solve numerical problems as literal problems first, and introduce numerical

[1] From a lecture delivered at the Institute for Teachers and Professors of Mathematics, held in August, 1959, at the University of Rhode Island.

values after the required magnitudes have been expressed in terms of the symbols of the given.

d) Discuss the expressions of the required magnitudes with the object of checking them, and of obtaining special or otherwise interesting cases.

e) Impress upon the student that the object of solving a hypothetical problem is to carry out a chain of reasoning which leads to the answer in a logical and orderly way, and brings out the underlying principles—that the purpose is not merely to find the answer.

f) Train the student to rephrase a verbal problem in its bare outlines and to form a strategy before starting the operation of solving it.

g) Whenever possible, denote geometrical and physical magnitudes by single letters, preferably by the first letter of the name of the magnitude.

h) In applied problems see to it that the terms of an equation have the same dimensions.[2]

Arithmetic. Any program designed to make mathematics attractive should begin with arithmetic, for the student's reaction against mathematics starts in elementary school.

Three years ago the principal of an elementary school came to see me. He asked me if I could do something about a group of bright fifth and sixth graders who were two years behind in arithmetic compared with their work in English. I met with fourteen of these children, six times. During the first meeting it became perfectly evident that they were not doing well in arithmetic because they were bored. I was appalled to learn that arithmetic is taught for eight years, using eight different books, and after examining the texts used in the fifth and sixth grades, I was not at all surprised that they were bored.

I should like to make three suggestions which should help to make arithmetic more attractive:

1. Do not require the solution of a great many single operation problems. The solving of such problems is comparable to learning how to button one's coat. Constant repetition is not necessary in either case.

2. Introduce problems which involve observation and measurement, as well as counting.

3. Introduce the rudiments of algebra into the lower grades, say the fifth grade.

The solution of an equation such as $x+2=5$ is no more difficult that that of the problem, "I hold five pennies in my hands; two of the pennies are in my left hand, how many are in my right hand?" In fact, this problem could be presented by actual pennies in the hands of the teacher as a preliminary to the solution of the corresponding equation.

During my sessions with the fifth and sixth graders, I gave a number of examples of the uses to which they could put their knowledge of mathematics. One of these was the determination of the height of a tall object, such as a flagpole, a tree, or a building, by measuring its shadow and that of a yardstick held upright. I introduced the concept of similar triangles and the proportionality of their sides. Those fifth and sixth graders understood what was being done in spite of the fact that they had been doing very poorly in arithmetic. It is important to note that I used H, h, L and l to denote the magnitudes involved instead of the mysterious x, x', y, y'. Such a problem as an actual project for pairs of children would be not only interesting to the children, but would give an opportunity to the teacher to explain some of the precautions which have to be taken in order to make the determination of the required height fairly accurate.

Algebra. Most of my suggestions of a general character are applicable to the presentation of algebra. I want to stress here the importance of solving verbal nu-

[2] H. M. Dadourian, *How to Study—How to Solve* (Reading, Mass.: Addison-Wesley Publishing Co., 1957).

merical problems as literal problems first. When a verbal numerical problem is solved literally, all problems of the same type are solved, irrespective of differences in the nature of the magnitudes and their numerical values. This is the main characteristic of algebra which distinguishes it from arithmetic.

I have found that college students act as if the main object of solving a numerical problem is to obtain a numerical answer, whether the problem is in physics, mechanics, or any other subject. They plunge into arithmetical operations, and when an answer is obtained, they follow it with the designation "Ans." and stop.

With the introduction of the elements of algebra in lower grades, it becomes practicable to include some of the topics of college algebra in a tenth grade course.

Geometry. I should like to see plane geometry given in the first semester of the eleventh grade, with its scope reduced to explanation of the mathematical concept of truth and illustrations of inductive reasoning by means of the proofs of a relatively few selected theorems. I would drop solid geometry from the curriculum. I am taking for granted that mensuration and presentation of solids on a plane would be taken care of in connection with courses in arithmetic and algebra.

Trigonometry. I should like to see trigonometry taught in the second semester of the eleventh grade. I would include in it a chapter on the addition and subtraction of vectors and their application to physics. In a text of only 98 pages, I have devoted a chapter to these without sacrificing any of the essentials of plane trigonometry. I was able to do this by such devices as replacing multifarious reduction formulas with a very simple rule of action.

Analytics and calculus. College students avoid these subjects whenever possible, in spite of the fact that these are the most interesting and, after arithmetic, the most useful subjects in mathematics. This is due partly to the distaste for mathematics engendered in school, and partly to the way these subjects are presented. Textbooks on analytics contain few, if any, applications to physics. Analytics can be made more attractive by such applications as that of the parabola to parabolic reflectors and of the hyperbola to sound ranging.

The calculus can be made more attractive by taking the mystery out of such concepts as limit and derivative, by an approach to these concepts along the lines indicated by suggestions 1 and 2 of general character.

For example, the concept of derivative could be approached in text or classroom as follows:

You have learned in elementary physics that the average speed of a body is given by

$$\bar{v}=\frac{s}{t}.$$

This equation does not tell the whole truth. In an automobile trip, for example, you do not measure the distance directly. As to the time, there is no way of measuring it directly. What you do is to take the reading of the odometer of your car, and the reading of a watch, at the beginning and at the end of the trip, and use the formula

$$\bar{v}=\frac{s_2-s_1}{t_2-t_1}.$$

This equation tells, however, more than the whole truth; for it gives the impression that every one of the symbols in the right-hand member is important *per se*, which is not the case. What is important is the change in the readings of the instruments. Denoting the changes in the readings by Δs and Δt, we obtain

$$\bar{v}=\frac{\Delta s}{\Delta t},$$

which tells the truth, the whole truth and nothing but the truth.

However, if you have an argument with a telegraph pole about your speed, the important question is not your average speed, but the speed at the instant of impact. But what is speed at a given instant? We may state that whatever it is, it is approached closer and closer by the average speed if the latter is determined for a very short interval of time before the instant, and the interval is made shorter and shorter; for the speed of a body cannot change abruptly unless the body hits a massive obstacle. Thus the speed of a body may be defined by the equation

$$\bar{v} = \lim_{\Delta t \to 0} \frac{\Delta s}{\Delta t}.$$

The transition from this equation to the definition of the derivative of a function is relatively simple.

Mathematics has been called the queen and the handmaiden of the sciences. A great majority of students feel at greater ease when she appears in the garb of a handmaiden than in the regal finery of a queen. My final suggestion is, therefore, to present her as much as possible as the handmaiden of the sciences.

Will you contribute to a forthcoming yearbook?

The Board of Directors of the National Council of Teachers of Mathematics at the April 1960 meeting approved the preparation and publication of a yearbook to be devoted to the problem of the mathematical education of the talented student in grades K–12.

It is intended that the yearbook be a *sourcebook* of topics and materials which have been found useful in enriching the mathematics program of talented students, but which are not parts of either traditional or experimental courses.

Under the chairmanship of Julius H. Hlavaty, the editorial committee, some members of which are listed below, has accepted the responsibility of preparing the grade level sections as indicated here. If you have material which you believe would make a contribution to achieving the purpose of the yearbook, won't you send it to the appropriate member of the committee, or to the chairman? Please submit your contribution no later than February, 1961. Where grade levels as listed below overlap, send the material to either member, but not to both. Full credit will be given to each contributor whose material is used.

K–8—Vincent J. Glennon
Director, Arithmetic Center
Syracuse University
Syracuse 10, New York

7–10—Joseph L. Payne
3019 University School
University of Michigan
Ann Arbor, Michigan

9–11—Henry Syer
Kent School
Kent, Connecticut

12, Honors—Harry D. Ruderman
Hunter College High School
930 Lexington Avenue
New York 31, New York

Julius H. Hlavaty, *Chairman*
Commission on Mathematics, CEEB
475 Riverside Drive
New York 27, New York

Teaching of mathematics in Switzerland

G. PÓLYA, *Stanford University, Stanford, California.*

A description of programs, teacher education, performance, and change in mathematics teaching and learning in Switzerland.

IN THE LAST years, my teaching activity has been almost entirely devoted to additional training of teachers of mathematics and science. Participants in my classes often asked me questions about European education, and so I gave a few talks, outside class time, about this subject, with some aspects of which I am well acquainted from personal experience. I wanted to use a few months' trip to Switzerland to gather additional information, especially about recent developments. I am greatly indebted to the Executive Committee of the Mathematical Association of America and to the U. S. Commission on Mathematical Instruction, which designated me their official representative to visit Swiss secondary schools to gain an up-to-date picture of current trends in mathematical education in Switzerland.

The present report summarizes my impressions. It is so written that most of it can be submitted to any group of high school teachers or interested parents and can be read by people who have no time or patience to absorb many details or many unfamiliar terms.

SUMMARY

In such complex human matters as the one before us, simplifications and generalizations are never very safe. Yet one cannot summarize without simplifications and generalizations. I am keenly conscious of all this when, in comparing the Swiss schools *with the average American high school*, I find the following four points the most significant:

I. Division into specialized branches. After primary school, that is, on the average from the seventh grade on, the school is divided into several specialized branches, and the curriculum within each branch leaves no choice, or little choice, to the student.

II. Teacher training emphasizes mastery of the subject matter much more, practice-teaching more, and general didactical lectures much less.

III. Performance. In those branches of the Swiss schools that prepare for University studies the programs in mathematics (and also in science and languages) go considerably further.

IV. Change. No major change has taken place in the recent past, and no major change is likely to take place in the near future, although there is some gradual change, some discussion, and some experimentation.

The foregoing refers to mathematics as taught in an average Swiss school compared with an average U. S. high school for boys and girls of the same age. Yet, in fact, points I, II and III (not point IV) are approximately correct to a much further extent: the teaching of mathematics, science, and languages in Central European schools shows on the whole the same contrast. By the way, all this is no news; it can be seen by any not quite casual and somewhat open-minded observer; it has been stated time and again —what is needed here is not so much fact-finding but rather fact-facing.

The following four sections illustrate the four points by adding appropriate details. An appendix says a few words about the personal experience of the author on which his opinions are based.

I. Division of the School Into Specialized Branches

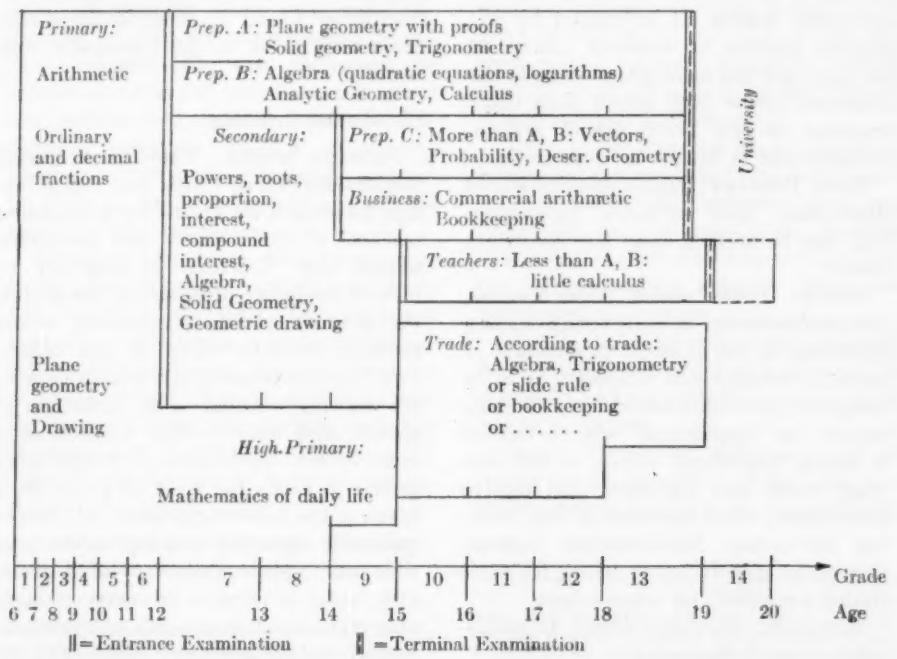
All the children (the future medical doctor, the future salesman, the future housewife, and so on) go to the same primary school, but eventually they must study different things, and so their studies must be separated from a certain point onward: the question is when and how. The basic difference between the European and U. S. school systems concerns this When and How.

The accompanying diagram "The Zurich School System" represents the system of public schools in the Canton of Zurich. (There is some simplification; more complex and less typical details are neglected.) I believe that it is a fairly representative example; any school system in Central Europe would show the same kind of contrast with American high schools.

The diagram shows clearly enough, I hope, how the school splits successively into several branches; it shows the number of years the students have to spend in each branch, and the students' ages; it shows moreover (sketchily) the subjects covered by the mathematical instruction in each branch. It emphasizes grades 7 through 13, gives less space to the first six grades, says nothing about the university level. I wish to explain the abbreviations and add a few remarks. (I add, in parentheses, the original German terms, and the corresponding terms used in the French-speaking Swiss cantons.)

PRIMARY: Primary school (*Primarschule, école primaire*) is for all children; it lasts six years. The separation into several

THE ZURICH SCHOOL SYSTEM


Primary:
Arithmetic
Ordinary and decimal fractions
Plane geometry and Drawing
Prep. A: Plane geometry with proofs
Solid geometry, Trigonometry
Prep. B: Algebra (quadratic equations, logarithms)
Analytic Geometry, Calculus
Secondary:
Powers, roots, proportion, interest, compound interest, Algebra, Solid Geometry, Geometric drawing
Prep. C: More than A, B: Vectors, Probability, Descr. Geometry
Business: Commercial arithmetic Bookkeeping
Teachers: Less than A, B: little calculus
Trade: According to trade: Algebra, Trigonometry or slide rule or bookkeeping or . . . . . .
High. Primary:
Mathematics of daily life
University
1 2 3 4 5 6 7 8 9 10 11 12 13 14 Grade
6 7 8 9 10 11 12 13 14 15 16 17 18 19 20 Age
‖ = Entrance Examination
▮ = Terminal Examination

branches begins after it, with the seventh grade.

PREP. A, B, C: Preparatory schools; they prepare the pupils especially for University studies.

Prep. A: Preparatory school of type A (*Literargymnasium, gymnase littéraire*) with Latin, Greek, and one living foreign language (French). Mathematics is taught in all grades through 6½ years.

Prep. B: Preparatory school of type B (*Realgymnasium; gymnase*) with Latin (no Greek) and two living foreign languages (usually French and English). The program in mathematics is the same as for type A.

Prep. C: Preparatory school of type C (*Oberrealschule, gymnase scientifique*). No Latin, two living foreign languages, more mathematics and science than in types A and B. It prepares principally for the study of science and engineering on university level.

SECONDARY: Secondary school (*Sekundarschule, école secondaire*). It prepares principally for professions which need no university studies. It is entered by the greatest fraction of students (about 60 per cent) and has more resemblance to the American junior high school than other branches of the Swiss school system. Mathematics is taught in each grade.

HIGH. PRIMARY: Higher primary school (*Oberschule, école primaire supérieure*). Still less theoretical than the secondary school.

TRADE: Trade school (*Gewerbeschule, école professionelle*). It has several branches, according to the different professions, of varying duration and intensity. It is an obligatory part-time school (8 to 10 hours weekly) for "apprentices" who are trained in shops, workshops, offices, or factories (they work also part-time and receive some salary, which increases as their training advances). Mathematical instruction varies greatly: quite strong for some trades, very little for some others.

BUSINESS: Business school (*Handelsschule, école de Commerce*).

TEACHERS': Teachers' preparatory school (*Unterseminar, école normale*). See Section II.

Entrance examinations are marked in the diagram by a double vertical line. For example, a pupil who has completed the primary school and wishes to enter the preparatory school of type A has to pass such an examination; he can enter the higher primary school without such an examination.

Terminal examinations which entitle the student to enter the university, or some branches of it (*Maturitätsprüfung, maturité*) are marked in the diagram by a triple vertical line (the middle line is dotted).

There is *full-time obligatory schooling* for 9 years to the age of 15 (in some cases only 8 years). Moreover, there is additional *part-time* obligatory schooling to the age 18; apprentices fulfill this obligation by attending trade school.

(The diagram shows also some less characteristic and less important details: the preparatory course of type C begins later than types A and B; some Business students may go on to the University, and so on.)

II. Teacher Training

PRIMARY SCHOOL. There is a special preparatory school (with two living foreign languages, no Latin) for prospective teachers of the primary and secondary schools (see "Teachers" in diagram). In view of the future profession of the pupils, the program of this preparatory school contains subjects which are not obligatory in other preparatory schools (music, art drawing, manual skills, technique of speech) and teaches other subjects to a lesser extent. Mathematics is taught in all grades (5 hours per week in grade 10, 4 hours in the following grades): Algebra to quadratic equations and logarithms (also slide rule), plane geometry, solid geometry, some descriptive geometry (acceptable representation of solids on the blackboard) analytic geometry, differential cal-

culus up to easy problems in maxima and minima.

Only the last year of this preparatory school brings some distinctively professional instruction (4 hours weekly in one semester, 3 in the other) to the prospective teachers, and this instruction is mostly practical: visits to schools and practice teaching with discussion following. General didactical ideas presented in class are closely tied up with the practical exercises.

The teachers' preparatory school is followed by the *one year teachers' training school on university level* (*Oberseminar*) which, again, is predominantly practical: lectures on general didactics, history of pedagogy, psychology and child psychology take 5 hours weekly out of 34 in one semester and $8\frac{1}{2}$ hours out of $32\frac{1}{2}$ in the other. The special didactics of various subjects are strongly practical; exercises are emphasized.

The main emphasis is on *practice teaching:* each candidate teaches 3 hours per week through the whole year under the guidance of an experienced teacher. Moreover, between the two semesters, he teaches 3 weeks full time in a city school and another 3 weeks in a country school. Having passed the training school, the candidate must teach successfully at least 40 weeks in the following two years to receive his final certificate.

SECONDARY SCHOOLS. The prospective teacher in a secondary school must have the primary teacher's final certificate and two years practice in a primary school, after which he receives *two years training at the university.* The program of this training contains didactical lectures, but the emphasis is on the mastery of the subject.[1] There are six branches of science teaching, divided into two groups:

1) Mathematics, Physics, Chemistry,
2) Botany, Zoology, Geography.

Any secondary teacher of science must take two subjects from group 1 and two subjects from group 2, do practice teaching in both groups and, in his final examination, teach a class for one hour in group 1 before the examiners, and another class in group 2. One of the branches chosen by any science teacher must be mathematics, and the minimum requirement is a course in calculus.

PREPARATORY SCHOOLS. A major number of the mathematics teachers for preparatory schools in all Swiss cantons receive their training at the Swiss Federal Institute of Technology (E.T.H., *Eidgenössische Technische Hochschule, École Polytechnique Fédérale*) in Zurich (a highly-rated technical university).

The program for the final certificate (mathematician's diploma) demands solid knowledge both in classical and in modern branches of mathematics; there are also courses and examinations in mechanics and in experimental and theoretical physics. An extensive paper on a special mathematical subject, for which three months' work is allowed, is also part of the final examination. In short, a mathematician's diploma at the E.T.H. is roughly equivalent to a strong M.S. degree.

About 250 semester-hours' work is needed for the mathematician's diploma;[2] only 7 additional semester-hours are needed for the additional pedagogical examination which is also required for teaching in preparatory schools. These are distributed as follows:

General didactics, 3 semester-hours: most of the time is spent on discussing the established school system, its purpose, its practical aspects.

Topics from school mathematics, 2 semester-hours: discussion of the most important topics which should be viewed dif-

[1] For instance, language teachers must have lived at least 6 months in a place where the language is spoken, and speak it fluently.

[2] A student takes about 30 hours per week, several of which, however, are devoted to practice sessions or to laboratory work. For instance, freshman calculus is taught in 9 hours weekly: 6 hours lecture, 3 hours practice sessions. As a result, there is relatively less homework than in an American university.

ferently at the university and in the preparatory school.

Practice teaching, 2 semester-hours: either all candidates listen to a class conducted by an experienced teacher, or one candidate teaches and the others observe. Each hour of practice teaching is followed by a discussion (particularly useful, if the instructor is experienced).

In the final examination, the candidate must teach twice, one hour each time, in the presence of the examiners.

Before the present teacher shortage (which began in Switzerland relatively late, two or three years ago) the (unwritten) requirements for a preparatory school teacher were even higher: a Ph.D. degree in the better paying city schools.

The teacher in a preparatory school has some advantages over the teacher in a secondary school: less teaching load, more salary, a little more security, but the differences are not so great; the salaries are in the proportion 5 to 4, or thereabouts. Yet the preparatory school teacher can do research and not infrequently may advance to a university position. (A personal note: among my former students who wrote their Ph.D. dissertations at the E.T.H. under my guidance, three are now professors of the E.T.H.; all three taught several years in preparatory schools.)

III. Performance

It is easy to compare programs, but it is not easy to judge the performance. For instance, there is a subject which is treated in Switzerland even in secondary schools, but which is conspicuously (and for the outsider, surprisingly) absent from the program of the average American high school: divisibility and the decomposition into prime factors. Yet to compare in performance the primary schools on both sides of the Atlantic, or the Swiss secondary schools with the junior high schools, is not so easy. At any rate, I am not in a position to undertake such a comparison.

There is, however, a point where the comparison is possible: the young people who pass the terminal examination (*Reifeprüfung, Maturitätsprüfung, maturité*) of a Swiss (or German, or Austrian, etc.) preparatory school, are about two years ahead of those boys and girls who, having finished high school, enter the university: they are about the same age and attended school the same number of years, but the boy with the "*Maturitätsprüfung*" can be admitted to an American university as a junior,[3] whereas the boy from the high school will be a freshman. As far as I can judge on the basis of my extensive experience on both sides of the Atlantic, this difference of two years persists: the boy who gets a B.A. or a B.S. after four years of study at one of the American universities I know intimately, is about as far advanced as the boy who, after two years of study, passes his intermediate examination at the E.T.H.

To see the full picture, we must realize that the minority of young people who pass the terminal examination (*Reifeprüfung, maturité*, etc.) of the Swiss, or European, preparatory schools (*Gymnasium, lycée, liceo*, etc.) results from a *selection*. To begin with, only a minority of children is able to pass the entrance examination into the preparatory schools. Of those who, having surmounted this obstacle, enter the preparatory schools, a considerable percentage will be dropped along the road because of insufficient achievements. (In Zurich, of 100 children entering the preparatory school, about 50 succeed in arriving at, and passing, the terminal examination.) Thus the preparatory school acts as a selecting agent, as a kind of sieve: it picks out the students who can be expected to deserve university education.

(It is obvious that any such procedure of selection, however beneficial it may be on the average, unavoidably involves hardships and undesirable results in some cases. It must be mentioned that young

[3] This is, to some extent, actual practice as far as I can judge from the cases I know personally.

people who do not succeed in the preparatory school have a second chance: they can attempt the "terminal examination" given by the state in regular intervals outside, and independently of, the preparatory schools [*Eidgenössische Maturitätsprüfung, maturité fédérale*]. There are several private schools—some of them are rather good—preparing people for these state examinations.)

Some readers of this report may raise the question: How is the performance of the teachers in preparatory schools affected by the circumstance that only a very small percentage of their training has been devoted to didactical subjects? (*Cf.* Section II.) This question was often in my mind when I spoke with teachers, but I met just one teacher who was really interested in it and had a decided opinion. Here it is, in a nutshell: "As a student at the university, I was impressed by some of my teachers. Then, when I started teaching in the school, I had difficulties. The trouble was, as I see it now, that I imitated my university professors: I did not realize that teaching in a school must differ in many ways from teaching at the university. The generalities I heard in our short course of general didactics gave me no help. I imagine that a good didactical course by a competent professor could have helped me to understand the situation sooner." Let us hope that my interlocutor was right.

IV. Change

In this section, I speak only about the preparatory schools.

The program of the "state terminal examinations" (*Eidgenössische Maturitätsprüfung;* see Section III) is fixed by a federal ordinance (of 1925). This program provides a minimum standard for the schools: the materials on this federal program are included in the curriculum of the schools, by law or by consent.

A committee of the Union of Swiss Teachers of Mathematics and Physics recommended a carefully phrased new minimum standard for preparatory schools in 1958, with the avowed purpose that the federal ordinance of 1925 should be correspondingly modified. The recommended changes are by no means revolutionary; here are a few.

For all three types, A, B and C: Less time should be spent on computations with logarithms, and the slide rule should be extensively used.

For types A and B: Compound interest and especially annuities should be dropped and the elements of the calculus introduced.

For type C: Spherical Trigonometry should be dropped, and several other things introduced, especially vectors; geometric transformations should be treated and so treated that they may serve as a preparation for the concepts of group theory.

These recommendations have been adopted by the schools to a good extent (most of them by the better schools, I am inclined to believe). At any rate, these recommendations have been widely discussed. There is a lively discussion and a strong interest in some schools, but the changes envisioned are by no means subversive. For instance, in the program of a school, mathematics and science gained a few semester-hours at the expense of Latin. Some teachers would like to teach a little more probability and add some statistics; others would insist more on geometric transformations and the group concept; and so on. In several schools, the programs are sufficiently elastic so that the teachers have a possibility of putting their ideas into practice.

Some teachers in the French-speaking Swiss cantons, especially in Geneva and Neuchâtel, would go much farther; under French influence, they would like to "Bourbakize" the curriculum. In fact, they have started to do so to some extent or will start next year. There have been a a few clashes; the "progressives" charge the others with "stabilism." I give the floor to one of the "stabilists" who explained to me why he objected so strongly

to a talk by a "progressionist": "He need not tell us what a set, a ring, a field or a mapping is, we have heard all that and more at the university. He should have told us how and when, with which examples, for which use, he introduced, or intends to introduce, such concepts, and how much resistance he has encountered from the students."

I refrain, of course, from taking sides. I am inclined to believe that some "Bourbakism" will penetrate into the Swiss schools eventually, by degrees. On the other hand, it seems to me that the Swiss preparatory schools function, on the whole, satisfactorily, and so some "stabilism" is justified. Experience and precaution are very much needed when new concepts or methods are introduced at the expense of tried and tested ones.

Appendix

Opinions such as have been voiced on the preceeding pages cannot be "proved": they necessarily depend on the experience and the background of the author. Therefore, the reader is entitled to know through what experience the author has acquired his opinions.

I attended a school (in Budapest, Hungary) similar to the Swiss preparatory school of type A. I did practice teaching for a year in a school (also in Budapest) similar to the Swiss preparatory school of type C. In the Zurich preparatory schools of types A and B, I was part-time teacher for about three years around 1920. Later, for more than ten years, I was in charge of the mathematical part of the admission examinations to the E.T.H. (see Section II) and there I examined candidates coming from schools all over Europe.

Beginning in the fall of 1958 I spent about eight months in Switzerland. I visited several schools both in the German-speaking and the French-speaking parts, gave talks to pupils and teachers, and discussed their problems with the teachers (many of them attended my classes at the E.T.H. or I was well acquainted with them otherwise). I spoke also with some key people in charge of the mathematical training of future primary and secondary teachers.

Pieces of eight

The figure which represents the numeral eight (8) has several marvelous aspects. By it, one can demonstrate that half of 8 is zero, or 3, depending on how it is cut. And $\frac{3}{4}$ is 9, or 6, but the latter is correct, and therefore is not acceptable for this essay. In Roman numerals, one can twice get 100 as $\frac{1}{2}$ of 8. Another interchange shows 8 to be a half of XIII, but the reverse cannot be managed. It is scarcely proper to mix things such as by VIII$\frac{1}{2}$. (Incidentally, in Roman numerals one can take 1 from 4 and get 5. And speaking of the Romans, who never conquered Scotland, that great architect McMix would never allow his name to be put on buildings for which the cornerstone was laid in 1909.)

Eight is a working number, as shown by octaves in music, eighths in the stock market, and 8-cylinder motors. Then there *was* the 8-hour day. Then too, skating and bandage tying involve the figure eight. Thus, the lucky number 7 is bested by work, and even in its own field by the 8th wonder of the world.

Eight is also one of the hole numbers, which includes 0, 6, and 9, but it is holier than the others.—*Gerald J. Cox.*

Problems in presenting non-Euclidean geometries to high school teachers

LOUIS O. KATTSOFF, *Boston College, Boston, Massachusetts.*

What is the place of non-Euclidean geometries and miniature systems in the secondary school curriculum, and in the curriculum for prospective secondary school teachers of mathematics?

I. Introduction

THIS PAPER contains some observations and impressions formed during a course in which I presented the basic notions of non-Euclidean geometries to high school teachers. The course was divided into three parts. The first was an introduction to the theory of axiom sets and, in particular, miniature geometric systems. The second was an analysis of Euclid during which the need for the Dedekind postulate, Pasch's axiom, and other assumptions was demonstrated. In addition, the existence of logical equivalents to the fifth postulate was noted with special attention directed to Playfair's axiom. The third and final section of the course showed how to contradict Playfair's axiom and developed some basic theorems of hyperbolic and elliptic geometry. The question of the consistency of Euclidean geometry was discussed and related to these alternative geometries. The notion of a projective geometry was briefly considered.

One aim of the course was to see how much, if any, of the material covered could be used in the secondary schools to which these teachers would return. If the teachers were to introduce non-Euclidean concepts in their own programs, one would suppose that their reactions and attitudes would be reflected in those of their students. Indeed, it goes without saying that they would impart these attitudes and reactions to many of their students. As it turned out, the kinds of difficulties many of the teachers had in grasping the basic ideas of deductive systems and alternative geometries were precisely those that I would expect their students to have. I believe this paper may be of benefit to those teachers preparing to present some non-Euclidean concepts to their students.

II. The deductive character of geometry

Although there is increasing emphasis on the nature of a proof and the structure of a geometry, the abstract character of a geometry is not always apparent to the student. That the axioms are merely the starting point for the deduction of theorems and in consequence can be modified and replaced at will is apparently more difficult for the student to grasp than is realized. The difficulty is the tendency to assume that axioms must somehow be "true." It must be stressed that the geometric system is merely saying "If our axioms are consistent, then the following theorems are valid." If the student once gets the idea that mathematical systems are completely abstract and provide us only with the consequences of a set of accepted premises, then the possibility of alternative geometries becomes significant.

The existence of alternative geometries is no longer a complete secret to high

school mathematics teachers and students. This, however, is merely to say that they are mentioned as existing—a kind of curiosity. The logical import of these alternatives is lost because usually nothing more is done with them. Only when the alternative axioms are stated and their consequences drawn, does the existence of non-Euclidean geometries have its full impact on the minds of teachers and students. The deductive character of geometry is then revealed completely.

The impact is increased if and when some consideration is given to the notions of proof and consistency. The distinction between truth and validity is clarified, and the student sees that although a set of axioms may be factually false or undecidable, or may even contradict some other set of axioms, it is still possible to deduce their consequences (theorems), provided the axioms are consistent. And the demonstration that certain non-Euclidean geometries are consistent if Euclidean geometry is consistent helps underline the logical nature of a geometric system and its independence of factual considerations.

III. The possibility of "other worlds"

Inevitably the question arises, "What geometry best describes the world in which we live?" Quite clearly this is not a mathematical question but, if anything, an empirical one. Yet I found that some discussion of this question changed attitudes and led to open-mindedness. It was possible to emphasize again that we were concerned with logical possibilities in mathematics, and that the term non-Euclidean geometries implies nothing derogatory to Euclid. Some of the teachers, particularly those who had been teaching plane geometry for many years, felt at first that a discussion of non-Euclidean geometries constituted a "tearing-down" of Euclid.

Apart from the important concept of logical possibilities, the equally important psychological effect of considering that "other types of worlds" were possible introduced a greater flexibility of imaginative creativity. This made it possible to clarify the concept of "space." The "receptacle" view of space is most common among teachers. It was accordingly difficult to get across the idea that, mathematically speaking, "space" could be viewed as simply a set of sets of numbers with certain relations defined. Once this notion was grasped, it was possible to introduce the concept of translation, that is, of associating the symbols of a geometry with words and expressions that pertained to "physical reality." Geometry could then be seen as an abstract system whose symbols were mapped to words like "point," "line," and "plane," *i.e.*, as an interpreted system, or as a mathematical model.

IV. The nature of a deductive system

One very interesting feature of the work on miniature systems was the revelation that few teachers in the class were aware of the nature and structure of a proof. Most had never become acquainted with the basic logical concepts underlying mathematical proofs. In our program there was a separate course on symbolic logic which correlated quite nicely with the course on non-Euclidean geometries and aimed to provide these teachers with material sufficient to close this gap in their knowledge. Let me list a few difficulties.

Although it was easy enough to define as contradictory two statements "p" and "q" such that when one is true, the other must be false, and to illustrate this definition with simple statements such as "The triangle ABC is isosceles" and "The triangle ABC is not isosceles," the teachers encountered great difficulties in writing the contradictories of compound statements, especially of conditional statements, such as: "If ABC is a triangle in Euclidean space such that the base angles are equal, then the sides opposite these angles are equal."

In the first place, it was necessary to show that the form of this statement is

$$t \cdot b \rightarrow s$$

where t represents "*ABC* is a triangle in Euclidean space,"
b represents "*ABC* has its base angles equal,"
s represents "The sides of *ABC* opposite the base angles are equal,"
$\cdot$ represents "and,"
$\rightarrow$ represents "implies."

Many teachers wanted to say that

$$t \cdot b \rightarrow s \text{ is false}$$

is the contradictory. Others preferred to say that the contractictory is

$$(t \cdot b) \text{ is false} \rightarrow s \text{ is false.}$$

Still others wrote as the contradictory

$$s \text{ is false} \rightarrow (t \cdot b) \text{ is false.}$$

The diversity of replies clearly indicated a fundamental misunderstanding of the logical concepts "contradictory" and "conditional." A few of the things that can be done to clarify these concepts are illustrated.

Suppose we define as follows:

The conditional "if p is true, then q is true" means "It cannot be the case that p is true and q is false." If we use "$\sim$" to denote "is false" then,

$$p \rightarrow q = \text{def } \sim(p \cdot \sim q).$$

In case p is compound, as in our example, then

$$t \cdot b \rightarrow s \text{ means } \sim[(t \cdot b) \cdot \sim s],$$

i.e., it cannot be the case that *ABC* is a triangle in Euclidean space and has its base angles equal and does not have the sides opposite these angles equal. The contradictory is obviously

$$(t \cdot b) \cdot \sim s$$

which may be read:

'*ABC* is a triangle in Euclidean space and has its base angles equal but it is false that the sides opposite the base angles are equal."

If now we introduce the operations of contraposition and exportation, students can be led to interesting insights.

Contraposition may be defined as:

$p \rightarrow q$ is equivalent to $\sim q \rightarrow \sim p$.

Exportation may be defined as:

$p \cdot q \rightarrow r$ is equivalent to $p \rightarrow (q \rightarrow r)$.

Now, returning to our example, we may calculate in the following fashion:

$$t \cdot b \rightarrow s$$
$$b \cdot t \rightarrow s$$
$$b \rightarrow (t \rightarrow s)$$
$$b \rightarrow (\sim s \rightarrow \sim t)$$
$$(b \cdot \sim s) \rightarrow \sim t,$$

i.e., if the base angles of a triangle are equal and the sides opposite them are not equal, then the triangle is not Euclidean. This leads to a recognition of one possible theorem in a non-Euclidean geometry of some sort.

The significance of the concept, *contradictory of a statement*, becomes apparent when one attempts to form the negation of Euclid's fifth postulate. The complex form of the fifth postulate in Euclid gave the students a good deal of trouble.

"If a straight line falling on two straight lines makes the interior angles on the same side less than two right angles, the two straight lines, if produced indefinitely, meet on that side on which are the angles less than the two right angles."

But even the statement, "Through a point outside a line, both in the same plane, one and only one line can be drawn parallel to the given line," was difficult. To negate this is to assert that it is false that one and only one line can be drawn through the given point under these conditions. The alternative statements are:

"No lines can be drawn parallel," etc.

"More than one line can be parallel," etc. Each of these is a denial but not a contradictory of the given statement.

The idea of a rigorous proof as a series of transformations is most difficult to put across because of the way geometry is taught. The analysis of some proofs in the traditional texts reveals the need for additional axioms. Such an analysis indicates the necessity for Dedekind's axiom, Pasch's axiom, and others. A study of Hilbert's axiom set for Euclidean geometry helped underline the need for the explicit recognition and statement of all the assumptions made. A number of teachers told me they were astounded at the actual lack of rigor in their courses when they supposed their proofs completely rigorous.

As one would expect, the idea of consistency and independence of axiom systems also raised difficulties.

V. Visualization of figures

Probably the most persistent difficulty was psychological. The habit of visualizing Euclidean figures kept constantly interfering with the grasp and comprehension of non-Euclidean statements and proofs. The relation of figures to theorems was widely misconstrued. Almost invariably when some figure was drawn for a theorem in non-Euclidean geometries, someone would say, "But the figure shows the theorem impossible." For example: when a Saccheri quadrilateral with summit angles obtuse (or acute) was put upon the blackboard, a number of teachers remarked, "But that cannot be because the summit must be parallel to the base!" With difficulty they were convinced that some imaginative reconstruction was necessary on their part, and that the theorem stated what *would* be the case if something else were the case. I must quote in this connection the remark of one teacher. After I had demonstrated a number of theorems in hyperbolic geometry, this teacher came to me and said, "I know that what you are doing is utterly impossible, yet there you are at the blackboard doing it."

Fundamental to these psychological difficulties is the ingrained belief that geometry (Euclidean) describes "figures in space" and hence, since a figure is what it is, Euclidean geometry alone can be true. This, incidentally, is precisely the point of view that made Saccheri set out to "vindicate" Euclid. Familiarity with Euclidean ways of conceiving "space" ends in the identification of familiarity with truth and obviousness. Some recital of the history of efforts to determine the "nature of physical space" in the nineteenth century and some explanation of relativity space helps the teachers break the bonds of Euclidean insularity.

VI. Concluding remarks

The tendency today is to overlook geometry and emphasize algebra. This is in large part due to the tremendous developments in algebra and their applications to computers. The expression "modern mathematics" is almost identified with modern algebra, statistical inference, and set theory. The consequent neglect of geometries as such will sooner or later be remedied.

Apart from other considerations, the intrinsic beauty of non-Euclidean geometries makes a course in non-Euclidean geometries particularly attractive. If the aim of the various programs for secondary school teachers of mathematics is to shake the teachers loose from traditional approaches and to show them what mathematical rigor is really like, then the study of non-Euclidean geometries is indispensable.

Finally, let me say a few words about the introduction of non-Euclidean concepts into the secondary school course in plane Euclidean geometry. A number of the teachers felt the material too difficult. Some of them had been teaching plane geometry for many years and were so deeply in a rut that they found it almost impossible to readjust their thinking.

Other teachers felt that many of the theorems were too difficult for their students to handle with success.

There was, however, considerable agreement that two units of work could be introduced easily and profitably at the seconddary school level. The study of miniature systems is intriguing and not too difficult. Moreover, miniature systems are small models of the larger geometries and serve as excellent tools for revealing the structure of a mathematical system. All students should undertake the study of a few very simple miniature systems near the end of their regular work.

The second unit centers around the Saccheri Quadrilateral. In going back to Saccheri's work, I was struck by the simplicity of most of the theorems. An adaptation of these theorems which I am preparing should not be beyond the abilities of the better students. Moreover, I believe these students would be quite excited by the strange possible "worlds" opened to their intellectual scrutiny.

Quite obviously, I believe a course in non-Euclidean geometries to be extremely valuable at the present stage of the attempt to improve secondary school mathematics.

What's new?

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Is solid geometry an entrance requirement for engineering schools?

VINCENT BRANT, *Board of Education of Baltimore County, Towson, Maryland.*

A survey to discover whether engineering schools which require solid geometry for admission will accept a fused honors course instead.

To FUSE or not to fuse—that is the question. Curriculum planners for geometry courses have been beset by this problem of fusing plane, solid and co-ordinate geometry, for it is inextricably tied to the requirement of solid geometry for admission to accredited schools of engineering. For many years engineering schools were loath to drop the solid geometry requirement because of the importance of the three-dimensional concepts in mathematics and engineering. At the same time mathematics teachers, supervisors, and curriculum planners have felt the impact of the recommendations of the Commission on Mathematics of the College Entrance Examination Board and have been anxious to update their programs. Yet they, too, have been reluctant to embark upon programs which may not meet with the approval of the colleges. The variations in admission requirements from college to college compound the problem for schools which wish to insure that their graduates will meet requirements for engineering schools. Many a workshop committee has been confronted with such an impasse.

The study which was stimulated by the efforts of a local workshop committee to provide a solution to this problem is described in this paper. The next pages deal with the problems met by the committee, the resulting study and a summary of the possible role to be played by geometry in the secondary school.

During the summer of 1959 a workshop in planning a curriculum for superior and gifted children, grades 1–12, was conducted by the Board of Education of Baltimore County, Towson, Maryland. The main effort of the committee in secondary school mathematics was the construction of a new geometry course which would follow the recommendations of the Commission on Mathematics of the College Entrance Examination Board by fusing the elements of plane geometry, solid geometry, and introductory co-ordinate geometry in a one-year course. In order to distinguish this course from the usual plane geometry course, our committee referred to it as honors geometry. With beginners' algebra and intermediate algebra offered in grades 8 and 9 respectively, the honors geometry course in grade 10 would eliminate the need for a semester course in solid geometry. This would permit a superior student to take semester courses in trigonometry, analytic geometry, and advanced algebra in grade 11, followed by a whole year of differential and integral calculus in grade 12. The transcript for colleges will list the new course as geometry with a notation explaining the content of the course.

The outline for this honors geometry course is as follows:

1. Introduction to Geometry
2. Direct Proof
3. Parallel and Perpendicular Lines and Planes: Related Angles

4. Proportion and Similarity
5. Co-ordinate Geometry
6. Quadrilaterals, Prisms, and Parallelepipeds
7. Circles and Spheres
8. Area of Plane Figures
9. Areas and Volumes of Solid Figures

It should be emphasized that the honors course fuses two- and three-dimensional concepts and elementary co-ordinate geometry wherever possible, as early as possible. Thus, the student avoids "compartmentalizing" branches of mathematics. The committee which developed this course was strongly against the kind of course which would take plane geometry as a block, superimpose solid geometry on the first block, and then cap it all with a unit on co-ordinate geometry.

The plan of the workshop committee to eliminate solid geometry as a semester course in this advanced placement program raised several questions which have confronted curriculum planners for years. These questions may be stated as follows:

1. What engineering schools still require solid geometry as an entrance requirement?
2. If a geometry course is constructed so as to include plane geometry, solid geometry, and co-ordinate geometry, will engineering schools which still require solid geometry accept this new geometry course in lieu of the required solid geometry course?

Part of the dilemma of curriculum planners was clarified in a study made by Dr. M. L. Keedy of the University of Maryland. The results of this study may be found in the February, 1959, issue of THE MATHEMATICS TEACHER. Dr. Keedy contacted 147 engineering schools in the United States which offer a first degree in engineering and are accredited by the Engineers Council for Professional Development. In this survey, engineering schools were asked whether or not they still required solid geometry as a requirement for entrance, and whether students would have to take a no-credit course in solid geometry in college if they had not taken it in high school. Of the 147 schools in the study, 96 schools indicated that they no longer required solid geometry for entrance, although 17 recommended it. The names of the 96 schools not requiring solid geometry were not listed in the summary given in THE MATHEMATICS TEACHER. The names of these schools have been made available to this writer through the kindness of Dr. Keedy and will be listed in a chart to be given later in this paper. In this former study, Table 2 listed the names of 29 schools which required a semester of solid geometry in high school or required that it be taken after admission into college. Table 3 listed the names of 9 schools which required solid geometry for admission at that time but were considering its elimination in the future. Table 4 listed the names of 13 schools which did not reply to Dr. Keedy's questionnaire.

Although the above survey gave our workshop a firm footing on which to proceed, it was felt that the 51 schools listed in Tables 2, 3, and 4 should be surveyed as to their policy in regard to the question:

Will a geometry course of one year including plane geometry, solid geometry, and introduction to co-ordinate geometry, as constructed in Baltimore County, be accepted in lieu of the solid geometry requirement?

The results of this survey of 50 schools* are presented in this paper, which gives in chart form the combined results of Dr. Keedy's study and this study. In order to make a distinction between the studies conducted by Dr. Keedy and the writer, two different letters will be used in the chart. The letter "K" will indicate the 96 engineering schools which do not require solid geometry, as given in Dr. Keedy's report. The letter "B" will indicate the schools contacted by the writer to ascertain whether or not our honors geometry course, fusing plane, solid and elementary co-ordinate geometry for superior and

* U. S. Naval Postgraduate School omitted from this study because high school students would not be making application to that institution.

Study	*Engineering School*	*Solid Geometry Requirement*	*Honors Geometry Acceptable*	*Preference*
K	Air Force Institute of Technology	No		
B	Akron, University of	Yes	Yes	Trigonometry or solid geometry
B	Alabama Polytechnic Institute	Yes	Yes, upon satisfactory completion of honor courses	
B	Alabama, University of	No	Yes, upon satisfactory completion of advanced placement program	
B	Alaska, University of	No	Yes	Requires entrance placement test
K	Arkansas, University of	No		
B	Arizona, University of	Not specifically required	Yes	1½ units algebra, 1 unit plane geometry, ½ unit trigonometry or solid geometry or additional algebra
B	Bradley University	No, but recommended	Yes	3½ units algebra, 1 unit plane geometry, ½ unit trigonometry
K	Brooklyn, Polytechnic Institute of	No		
B	Brown University	Yes	Yes	
B	Bucknell University	Not specifically required	Yes	4 years of college preparatory mathematics with trigonometry
K	California Institute of Technology	No		
B	California, University of	No	Not answered	2 units algebra, 1 unit plane geometry, ½ unit trigonometry, ½ unit mechanical drawing
K	Carnegie Institute of Technology	No		
B	Case Institute of Technology	Not specifically required	Yes	4 years of college preparatory mathematics with trigonometry
K	Catholic University	No		
B	Cincinnati, University of	Not specifically required	Yes	4 years of college preparatory mathematics
B	Citadel, The	No		2 units algebra, 1 unit plane geometry and CEEB (SAT) and achievement tests
K	Clarkson College of Technology	No		
K	Clemson A. and M. College	No		
K	Colorado A. and M. College	No		
K	Colorado School of Mines	No		
K	Colorado, University of	No		
B	Columbia University	No	Yes	Intermediate algebra and trigonometry
K	Connecticut, University of	No		
K	Cooper Union School of Engineering	No		
K	Cornell University	No		
B	Dartmouth College	No	Yes	
B	Dayton, University of	No	Yes	
B	Delaware, University of	No	Yes	
K	Denver, University of	No		
K	Detroit, University of	No		
K	Drexel Institute of Technology	No		
B	Duke University	No	Yes	4 years of college preparatory math.
K	Fenn College	No		
K	Florida, University of	No		
K	George Washington University	No		
K	Georgia Institute of Technology	No		
K	Harvard University	No		
B	Hawaii, University of	No	Not answered	2 years algebra, 1 year plane geometry, ½ year trigonometry
B	Howard University	No, but recommended	Yes, upon completion of the advanced placement program	
K	Idaho, University of	No		
K	Illinois Institute of Technology	No		
K	Illinois, University of	No		
K	Iowa State College	No		
K	Iowa, State University of	No		
K	Johns Hopkins University	No		
K	Kansas State College	No		
K	Kansas, University of	No		
B	Kentucky, University of	Yes	Yes	
K	Lafayette College	No		
K	Lehigh University	No		
K	Louisiana Polytechnic Institute	No		
K	Louisiana State University	No		

Study	Engineering School	Solid Geometry Requirement	Honors Geometry Acceptable	Preference
B	Louisville, University of	Optional	Under consideration	4 years of college preparatory mathematics required. Introduction to analytic geometry and calculus accepted in lieu of solid geometry.
K	Maine, University of	No		
K	Manhattan College	No		
K	Marquette University	No		
K	Maryland, University of	No		3½ years of college preparatory math.
K	Massachusetts Institute of Technology	No		
B	Massachusetts, University of	Yes	Yes	Requires trigonometry
B	Michigan College of Mining and Technology	Yes	Yes	Requirement of solid geometry under consideration
K	Michigan State University	No		
B	Michigan, University of	Yes	Yes	Transcript should show that the course fuses plane, solid and elementary co-ordinate geometry.
B	Minnesota, University of	Yes	No definite answer	3½ years of college preparatory mathematics. Content of courses most important. Uses a qualifying test.
K	Mississippi State College	No		
K	Mississippi, University of	No		
B	Missouri School of Mines and Metallurgy (Rolla, Missouri)	No	Yes	Recommends fusion course in geometry in its bulletin
K	Missouri, University of (Columbia, Missouri)	No		
B	Montana School of Mines	Yes	Yes	
B	Montana State College	Yes	Yes	All geometry courses, not only honors geometry, should fuse plane, solid and co-ordinate geometry.
K	Nebraska, University of	No		
K	Nevada, University of	No		
K	New Hampshire, University of	No		
K	New Mexico College of A. and M. Arts	No		
K	New Mexico, University of	No		
K	New York, City College of	No		
K	New York University	No		
K	Newark College of Engineering	No		
B	North Carolina State College	Under consideration by Department of Mathematics and School of Engineering		
B	North Dakota State College	Not answered		
K	North Dakota, University of	No		
K	Northeastern University	No		
K	Northwestern University	No		
K	Norwich University	No		
B	Notre Dame, University of	No		
K	Ohio Northern University	No		
K	Ohio State University	No		
B	Ohio University	No	Yes	
K	Oklahoma A. and M. College	No, but recommended		2 units algebra, 1 unit plane geometry, ½ unit trigonometry
B	Oklahoma, University of		Incomplete answer	
K	Oregon State College	No		
K	Pennsylvania State University	No		
K	Pennsylvania, University of	No		
K	Pittsburgh, University of	No		
K	Pratt Institute	No		
K	Princeton University	No		
K	Purdue University	No		
K	Rensselaer Polytechnic Institute	No		
B	Rhode Island, University of	Yes	Yes	Transcript should show that the geometry course content includes plane, solid and co-ordinate geometry.
K	Rice Institute	No		
K	Rochester, University of	No		
B	Rose Polytechnic Institute	No, but recommended	Yes	
K	Rutgers University	No		
K	Saint Louis University	No		
K	Santa Clara University	No		
K	South Carolina, University of	No		

Study	*Engineering School*	*Solid Geometry Requirement*	*Honors Geometry Acceptable*	*Preference*
B	South Dakota School of Mines	No	Yes	Advanced placement program acceptable
K	South Dakota State College	No		
K	Southwestern Louisiana Institute	No		
K	Southern California, University of	No		
K	Southern Methodist University	No		
K	Stanford University	No		
K	Stevens Institute of Technology	No		
K	Swarthmore College	No		
B	Syracuse University	No	Yes	2 units algebra, 1 unit plane geometry, ½ unit trigonometry. Advanced algebra strongly recommended.
B	Tennessee, University of	Yes	Not answered	Concerned with content of course, not with names of units.
K	Texas A. and M. College	No		
K	Texas College of Arts and Industries	No		
K	Texas Technological College	No		
B	Texas, University of	Yes	Yes	May eliminate solid geometry requirement
B	Texas Western College	Yes	Yes	
B	Toledo, University of	No	Yes	
K	Tufts University	No		
K	Tulane University	No		
B	Tulsa, University of	Yes	Yes, upon completion of superior and gifted program	
B	Union College	Not required specifically	Yes	3½ units of college preparatory mathematics
K	U. S. Coast Guard Academy	No		
K	Utah, University of	No		
K	Vanderbilt University	No		
B	Vermont, University of	No, but recommended	Yes	
K	Villanova University	No		
B	Virginia Military Institute	Yes		Qualifying test required for solid geometry credit
B	Virginia Polytechnic Institute	Yes	Under consideration	
B	Virginia, University of	No	Yes	2 units algebra, 1 unit plane geometry, ½ unit advanced algebra or solid geometry or trigonometry
K	Washington, State College of	No		
K	Washington University	No		
K	Washington, University of	No		
B	Wayne State University	No	Yes	2 units algebra, 1 unit plane geometry, ½ unit trigonometry strongly recommended. Requires placement test.
B	West Virginia University	Yes	Yes, if transcript indicates that this is an advanced mathematics program.	
B	Wichita, University of	Yes	Yes	Transcript should show the content of the honors geometry course.
K	Wisconsin, University of	No		
K	Worcester Polytechnic Institute	No		
K	Wyoming, University of	No		
K	Yale University	No		
K	Youngstown University	No		

SUMMARY OF SCHOOLS IN STUDY B

I.	Number of schools which do not require solid geometry or do not specifically require solid geometry and accept honors geometry	23
II.	Number of schools which do not require solid geometry but did not give an answer to the acceptability of honors geometry	4
III.	Number of schools which still require solid geometry, but accept honors geometry in lieu of solid geometry	15
IV.	Number of schools which still require solid geometry, but have not given an answer to the acceptability of honors geometry, or are *considering a change in requirements*	5
V.	Number of schools which still require solid geometry and did not specifically accept honors geometry but would use entrance placement test of their own making	2
VI.	Number of schools not answering	1
	Total	50

gifted children, would be acceptable in lieu of the usual one-year course in plane geometry and the semester course in solid geometry.

It is interesting to note that, of the 38 schools previously reported by Dr. Keedy as requiring solid geometry, 16 indicated that solid geometry is no longer a specific requirement for admission. Some schools had arrived at this decision reluctantly, and many are in favor of a geometry course which would fuse plane, solid, and elementary co-ordinate geometry. Fifteen schools which still require solid geometry for admission are willing to accept honors geometry instead. Five schools had referred the problem to their engineering or mathematics departments or were considering a change in the solid geometry requirement.

Of the 146 engineering schools covered in both studies, 123 no longer required solid geometry, though many recommended it. In additon, 38 of the 50 schools in study B would accept honors geometry in lieu of solid geometry. This total does not include 4 schools which no longer require solid geometry but which failed to give a specific answer to the acceptability of honors geometry. This report assumes that all 96 schools not requiring solid geometry in study K would also accept honors geometry, and therefore a grand total of 134 schools would accept a fusion geometry course. This assumption is reasonable because schools in study B which no longer required solid geometry generally accepted honors geometry.

The schools contacted in study B were most helpful and reacted favorably to our efforts to deal with this problem, of which they are acutely aware. Some of the comments in regard to this fusion type geometry are quoted below:

Columbia University: The course which you have outlined is an admirable one. The combination of plane geometry, solid geometry, and elementary co-ordinate geometry is obviously a good one and a program growing in popularity. You will be pleased to know, also, that the Columbia Engineering Faculty has voted full recognition to the Advanced Placement Program.

Dartmouth College: The Department of Mathematics at Dartmouth College has been in close contact with the Commission on Mathematics of the College Entrance Examination Board, and our faculty members are in firm agreement with the recommendations made by that particular group. I have discussed the contents of your letter with the Mathematics Chairman, and he is in full favor of the honors geometry course which you have outlined.

University of Delaware: We think that the newly-organized course is an excellent combination and want to compliment you upon this step.

Duke University: We are pleased to learn of your new program in geometry and congratulate you upon this progressive step. We feel that this course will meet our geometry requirements in every respect.

University of Massachusetts: I [Dean of School of Engineering] am very much interested in the outline of your so-called honors geometry course and I personally feel that this may very adequately cover the space concepts that a student normally gets in the solid geometry work. In most engineering curricula, as you know, the student does get a course in descriptive geometry which uses space concepts a great deal. He will certainly be better prepared for such work if he has had solid geometry in some previous course. If the three dimensional aspects of geometry are included in the geometry course, this should be quite adequate.

Missouri School of Mines and Metallurgy: Our school no longer requires Solid Geometry for admission into our engineering program. At the time the change was made our faculty recommended that students planning to enter this school complete a one year course in geometry while in high school which would cover exactly the subject areas included in your geometry course.

University of Virginia: I believe the program in geometry which you describe in your letter is excellent and would certainly satisfy our requirements in geometry.

University of Wichita: With respect to the course outlined in your letter I believe that there would be no question whatsoever about its acceptance if the transcript from a high school showed plane and solid geometry.

Commission on Mathematics: Thank you very much for the copy of your tentative guide for a course in geometry for the superior and gifted student. You and your committee deserve great praise for a first rate job.

What import do the results given above have for the curriculum planner in mathematics? The following probable conclusions may be summarized from the study:

1. The elimination of solid geometry as an entrance requirement must not be taken to

mean that the three-dimensional concepts are unimportant. The reverse is true. Many of the schools which do not specifically require solid geometry recommend its inclusion in the high school program. The fusing of two- and three-dimensional concepts is considered by many schools as a method of resolving their reluctance in eliminating the solid geometry requirement.

2. Curriculum planners should consider the revision of present geometry courses for *all* students to include two- and three-dimensional geometry. The course in honors geometry for superior and gifted students in Baltimore County was initiated as a first step to break away from the traditional approach to geometry. The committee hopes that after this experimental course has been in operation for a year or two, the resulting experiences will pave the way for writing a similar course for all geometry students. Such a course, however, would not have the depth which has been planned for the more able students. Such a fusion course for all students would have an added advantage since the traditional course in plane geometry is a terminal course in mathematics for many students. Thus, they would gain added insight into three-dimensional concepts and coordinate geometry before taking their final leave of mathematics. Many of the current high school textbooks reflect this trend and either incorporate two- and three-dimensional geometry or present brief units of solid geometry in supplementary pamphlets which accompany the plane geometry text.
3. Since many of the schools are granting full recognition to the advanced placement program, curriculum planners should consider the recommendations of the Commission on Mathematics of the College Entrance Examination Board in revising their programs in geometry.
4. Several schools recommended that high schools indicate on the transcript the content of such honors courses or fusion courses. The names given to many mathematics courses in high school have been misleading. For that reason, many engineering schools are interested in the content of such courses and would like to have the transcript indicate the contents or nature more fully.

The first reports by teachers of this course given at a quarterly meeting indicate that students are enthusiastic about it and think in terms of two and three dimensions with no difficulty at all. The teachers agree that this approach is stimulating for both teacher and student. Since this is a tentative course, it is likely that some modifications may be necessary, but it would seem at the present time that the current framework will remain unchanged. Our committee has the greatest praise for two publications of the Commission on Mathematics, the "Report" and the "Appendices." The two publications are highly recommended for curriculum planners.

It would seem that the trend towards fusion of plane, solid, and elementary coordinate geometry into one geometry course is gaining in popularity. The relaxing of the specific requirement of solid geometry as an entrance requirement to engineering schools will afford greater impetus to this new development. However, it must be emphasized again that the deletion of the solid geometry requirement does not mean that the three-dimensional concepts are unimportant, but rather that they are so vitally important that these concepts should be introduced in a revised geometry course.

Our colleges and universities must take part of the responsibility for the improvement of teacher-training programs. The staff for such programs must be carefully chosen. The teacher of a subject-matter course for prospective teachers should be, of course, an expert in his field, but he should have also a good understanding of the problems of secondary school teaching. Curricula need to be carefully planned, since it must be recognized that a subject-matter course designed for the training of a research scientist is not necessarily appropriate for the training of a prospective science teacher.—*E. G. Begle in* "Science Education *News*"

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The "Madison Project" of Syracuse University

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Although it operates in many schools—we do not, in fact, know how many—the Syracuse University "Madison Project" is probably one of the least well known of all of the "new" mathematics programs. This unintended secrecy derives, perhaps, from the nature of the project itself. As we shall see presently, the project is represented by neither a large collection of sample textbooks, nor a small collection of incisive slogans.

One theme: Building "readiness"

It is significant that the project is concerned with the study of mathematics in grades 4–9. One key—though by no means the only one—to the project's activities is a consistent emphasis on readiness. Now, by saying that a student possesses "readiness," the project means that the student has the necessary background skills, understandings, and motivation so that the lesson for the day has a chance to "click"—it will seem to the student that he has learned something new and interesting.

Readiness is perhaps most abundantly available in a class in English—ever since birth, a child has lived with an increasing desire to communicate and has come into daily contact with the basic components: he has heard spoken language and tried his own hand at talking; he has seen letters and words and sentences and punctuation marks on everything from a subway poster to the daily newspaper, or from a box of cereal to the screen of a TV set. He has been intimidated by inflections, and has likely experimented with attempts to intimidate others.

In short, he brings to the English teacher a complete set of component pieces, and now he has merely the task of learning to assemble them with increased discrimination: to distinguish one verb form from another, to select a comma when it is called for, or a semicolon if that is what is needed. . . .

When the student enters a ninth-grade algebra class, on the other hand, he brings with him an incredibly inadequate "readiness." Virtually nothing in his daily life has made him want to solve equations, and nearly every simple concept—such as truth set, or open sentence, or the classification of equations into linear, quadratic, and so on—and every single technique—such as using "placeholder" or "variable" notation, scrutinizing instances in an attempt to discover a pattern, combining statements according to an explicit system of inference, et cetera—are entirely new to him. In short order we propose to teach him what an *equation* is, what its component parts are (such as "variables" and "constants"), how one deals with equations in order to "solve" them, what we mean by a "solution," how to use equations in solving word problems, how to generalize to more complicated equations,

what sort of obstacles one may encounter and how to avoid them, and usually even a great deal more.

Do we succeed?

One would hardly think so, to see how we must reteach this material in grade 11, and—far too often—again in the freshman year of college.

Now, it is not the malicious resistance of children that demands this triple exposure. Quite the contrary, it is their (or our, for that matter) normal nature. We have to have some small acquaintance with a topic *before* we can formulate any sophisticated or profound questions about it, and we learn by a protracted pattern of misunderstanding, becoming aware of some of our limitations, deepening our understanding, extending the range of our curiosity—and then starting the cycle all over again.

Suppose a child were to appear in an English class, never having heard a word of spoken English, nor seen a word of written English, with no notion of "sentence" or "period" or "comma" or "letter of the alphabet," or how to read from left to right, or how to pronounce the English vowels, and so on. Would we need to teach the year's work again the following year? It seems virtually certain that we should, no matter how excellent our first year's presentation of the material may have been.

There is, after all, such a thing as expecting too much.

This, then, is the first task that the Madison Project has set itself: to study high school and college mathematics, determine what concepts and what techniques are most nearly essential, and to build readiness for them over a period of years, instead of over a period of days or weeks, as is normally done at present.

Thus far the project has focused primarily on ninth-grade algebra and is developing a program for grades 4–9 that tries to develop readiness for all of the really fundamental skills and understandings.

A SECOND THEME: WHAT IS MATHEMATICS?

If we look at this program in more detail, we shall find that, in analyzing the "essentials" of ninth-grade algebra, the project has again moved in a somewhat original direction.

The "essentials" of mathematics have not been taken to include, for example, the quadratic *formula*. Rather, the quadratic formula is regarded as a *result* of practicing mathematics. I may know the quadratic formula, and Cauchy's integral formula, and a hundred others, and yet not "know mathematics," in just the same way that I may own the largest collection of Vermeers in the state of Hawaii, without being able to paint.

The project has relegated specific "results" (such as the quadratic formula) to a kind of subsidiary status—they are the illustrations by which the true subject is presented. One might call them the trees which make up the forest, or—more appropriately—one might say that these specific *results* of mathematical research are like the individually insignificant blades of grass which together constitute an attractive lawn.

In the case of quadratic equations, what are some of the "essentials"?

Well, to begin with, there are the basic concepts of "open sentence," "placeholder" (or "variable" or "unknown"), and "truth set."

Then there is the matter of *classifying* open sentences into inequalities, equations . . . then *linear* equations, *quadratic* equations, etc. (Not that we mean here that a specific classification scheme is to be memorized—rather, there is some question to be raised about what *different kinds* of equations may possibly be encountered—and which are amenable to similar treatment, et cetera.)

Then there is the task of studying a variety of examples to see if we can discover a *pattern*. We do this much with third or fourth graders, who find that they can solve equations like

$$(\square\times\square)-(5\times\square)+6=0,$$

first by extended trial and error—getting the truth set $\{2, 3\}$—and then, once they notice (from many examples) the persistent *patterns*

$$\begin{cases}2+3=5\\2\times3=6\end{cases}$$

they can use this discovery to solve other equations more quickly: for example,

$$(\square\times\square)-(13\times\square)+22=0,$$

which yields $\{2, 11\}$, and so on.

At this point, we begin a more sophisticated level of operation, in which we try to answer such questions as these:

How would we *describe* our discovery to someone else?

Does the method always work? If not, can we extend it?

How can we use "placeholder" notation to write a single open sentence that will represent *every quadratic equation whatsoever?*

Are we *sure* that our final "formula" is correct? Always? *How* are we sure? (This last topic involves an extensive study of the sources of mathematical truth, and, in particular, the process of *inferring* theorems from a set of axioms.)

These, then, are some of the skills and concepts (and, one might add, some of the basic "culture" or "orientation") with which the ninth-grade student should come equipped, and which the Madison Project program therefore endeavors to build in grades 3–8.

The style of presentation

We mean, as it were, to *paint*, and not to collect reproductions of Picasso's paintings. Therefore, the student is confronted by a *problem*—this *problem* is the empty canvas, on which he is to make his mark.

The first questions, naturally, must be very simple, since the student will be expected to solve the questions *without help from us.*

"What number shall we write to get a true statement, if we are given

$$3+\square=5?$$

From this question we proceed to a long sequence of further questions, from which concepts and techniques will gradually emerge.

How shall we now write our textbook?

Evidently, we shall not be able to produce a text at all. A good art course is, perhaps, a sequence of experiences in filling up blank canvases, together with intelligent criticism and exciting stimulation. Just so with mathematics! Our "course" must combine many experiences in solving problems, seeking out patterns, and formulating concepts, in an atmosphere of discriminating and stimulating class discussion. (In place of a text, the project has prepared workbooks and teachers' manuals, and is now preparing tape recordings of actual project classes. The more experienced project teachers are, however, beginning to dispense with these, and take their cues from the class discussion itself.)

Much of the interest in the project's activities focuses, quite properly, on the style in which the classroom discussion is conducted, and this will be discussed at some length in various forthcoming publications.

Who teaches the classes?

Probably nearly every reader has by now recognized a central difficulty that must be overcome: If we are to build readiness for mathematics in grades 4, 5, and 6 (and sometimes in grade 3), and with a minimum use of a text and a maximum use of rather free-wheeling class discussions, then *who will do the teaching?*

Some elementary teachers are well able to do so, of course, but the number who can is not very large. In practice, the project's elementary school classes have been taught by elementary teachers in some cases, by secondary mathematics teachers in other instances, and in several cases by college mathematics professors.

Since this is a crucial problem, we may

find three types of solutions worth looking at:

1. *Specialist teachers.* In several communities, notably Syracuse, New York, and Weston, Connecticut, the project classes are being taught by specialist teachers.

In all cases where this is done, arrangements are made for other elementary teachers to observe project classes regularly, so that after a few years the school system is well provided with teachers who are familiar with the project material.

One great advantage that derives from the use of a specialist teacher is that, through frequent repetition of the same lesson to different classes, the teacher is able to achieve a very high proficiency in the handling of class discussions.

(The project classes mentioned may be visited by arrangement with Mr. William E. Bowin, Board of Education, Syracuse, New York, or with Mrs. Beryl Cochran, Weston Public Schools, Weston, Connecticut.)

2. *Mathematics clubs.* In any method, it is essential to obtain active co-operation between elementary teachers and secondary mathematics teachers. One of the easiest ways to implement this on a modest level is by means of mathematics clubs, meeting regularly one afternoon (say) every week. Several teachers—notably Mr. Henry Standish* of West Genesee Central High School, Camillus, New York—have made effective use of this method.

3. *Junior high school classes.* Of course, when time can be found in grades 7 and 8 to permit allocating one (or two) periods per week to project classes, there is likely to be much less of a problem in finding teachers to teach them, since most junior high school faculties can handle the material with relative ease.

In addition to these three methods of presentation, there is a different use of project material which might be mentioned—that of teaching high school children (usually in grade 9) who have poor records of mathematical achievement. Particularly effective use of project material with the weaker high school students has been achieved under the direction of Mrs. Eleanor Carney, of West Canada Valley Central School, Middleville, New York.

* Mr. Standish will describe his part in this project in the next issue of THE MATHEMATICS TEACHER.

THE THING ITSELF

The use of project material with weaker students gains some of its effectiveness from another project characteristic, the exceedingly sparse use of language. Not only do we minimize *exposition* (thereby greatly reducing the verbal demands of the course), but we *show*, and let the children themselves *describe* (and even the children's own verbalizations are not encouraged until we are sure of the "thing itself")

Mendelssohn once remarked that a musical theme is the least abstract thing in the world. It is what it is, and there is no word or symbol to "describe" it.

Fortunately—especially for those children who are not quick to respond to verbal communications—mathematics also has this property: we don't have to *say* what we mean, we can *show* what we mean.

Our earlier example with quadratic equations illustrates this. I can write (though we would never be so explicit in an actual class):

$$(\square\times\square)-(8\times\square)+15=0$$

$$\{3,\ 5\}\qquad 3+5=8\qquad 3\times 5=15$$

$$(\square\times\square)-(102\times\square)+200=0$$

$$\{100,\ 2\}\qquad 100+2=102\qquad 100\times 2=200$$

You can see the pattern itself! True, I *can* describe it, for example by saying that "the sum of the roots of a quadratic equation in normal form equals the negative of the coefficient of the linear term, and the product of the roots equals the constant term."

The important thing is that I *need not* describe the pattern in order to work with it. Even a low I.Q. student who would boggle absolutely at the phrase "the sum of the roots of a quadratic equation in normal form . . . ," can figure out, for himself, that if he wants an equation with the truth set $\{5, 2\}$, he can write

$$(\square\times\square)-(7\times\square)+10=0.$$

He *knows* what the pattern is, because he can *see* it.

The time to start talking is after we know what we are talking about. Once we have seen the thing itself, we can then attempt to describe it.

Relation to other programs

Like nearly all "new" mathematics programs, the Madison Project owes a great debt to Max Beberman of the UICSM Project, who (among his other contributions) proved the existence theorem that an improved program of school mathematics can be brought into actual use. The project approach also shows much of the influence of W. W. Sawyer. But even more important than acknowledging origins is the question of *compatibility*—can the Madison Project material be used to lead in to other "new" programs on the high school level, or, for that matter, into a good "traditional" program?

Obviously, it is too soon to cite any successful instances of this, but several schools have developed plans to use the Madison Project material to lead in to SMSG material (perhaps the SMSG "First Course in Algebra"), or into the UICSM material (at about grade eight). The project material can definitely be used to lead into a good "traditional" course. It also seems likely that the Madison Project material could effectively *follow* the Suppes arithmetic-by-sets (Patrick Suppes, Stanford University, Stanford, California), which is being planned for grades one and two. The Madison Project work is somewhat similar in spirit to the David Page Arithmetic Project (University of Illinois), and has also been influenced by the Mayor-Keedy UMMaP. It might be added that Madison Project experimentation to date has been supported by grants from the Marcel Holzer Foundation of New York City, and from the Carrier Corporation. Opportunities for teachers to study the project materials have been provided by grants from the National Science Foundation and the Alfred P. Sloan Foundation. The project, incidentally, is named after the Madison School, in Syracuse, N.Y., which was the site of the original project experimentation.

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Pennsylvania teachers participate in the NDEA in-service training program

Over 5,000 elementary teachers in Pennsylvania enrolled in 1960 in twenty science and four mathematics programs offered under Title III of the National Defense Education Act. The program has been in operation only six months. During the same period almost one thousand secondary school teachers enrolled in four science, six mathematics, and two modern foreign language programs. These teachers enrolled voluntarily with no college credit offered, on their own time.

• HISTORICALLY SPEAKING,—

Edited by Howard Eves, University of Maine, Orono, Maine

Why and how we should correct the mistakes of Euclid[1]

by Paul H. Daus, University of California, Los Angeles, California

I. THE WHY:

THE GLORY THAT IS EUCLID'S

"Euclid is the only man to whom there ever came or ever can come the glory of having successfully incorporated in his own writing all the essential parts of the accumulated mathematical knowledge of his time" [1].[2] He was the most successful textbook writer the world has ever known. Over a thousand editions or revisions of his famous *Elements* have appeared since the advent of printing, and his work has dominated the teaching of the subject ever since it appeared, first in manuscript form and then in the form of revised textbooks. Perhaps he was too successful, for two thousand years later, when new geometrical ideas and modern points of view were struggling for recognition, the unrivaled supremacy of Euclid and the mistaken belief in the infallibility of Euclid made the assimilation of these new ideas extremely difficult. Even today it is no easy task, and the School Mathematics Study Group, which is planning a different type of text for geometry, cannot be assured of any immediate success. The power of inertia must never be underrated; it often takes a crisis to arouse us from our lethargy.

Euclid unified the work of many scholars and systematized the known mathematics of the day. His outstanding contribution lies in his extended application of the mathematical method—the hypothetico-deductive method of modern mathematics. He set himself the task of finding an adequate and universally acceptable set of postulates for geometry and at the same time avoiding a proliferation of assumptions, many of which would be repetitious in the sense of nonindependence. He entertained the ideal of placing mathematics on an unimpeachably logical basis. He demonstrated how much knowledge can be derived by reasoning alone, and it was through his *Elements* that later civilizations learned the power of reason.

How well he succeeded must be expressed in the words of one of his later commentators, for all the original manuscripts have been lost. About 700 years after Euclid's *Elements* appeared, Proclus wrote, "The selection and arrangement of the fundamentals was complete, clear, concise, and rid of everything superfluous. The theorems were presented in general terms, rather than as a number of special cases, and in all ways Euclid's system was superior to all the rest."

[1] Presented to the Southwestern Section of the Mathematical Association of America, April 10, 1959.

[2] Numbers in brackets refer to References at the end of the article.

The discovery of non-Euclidean geometries

Euclid began his *Elements* with a systematic arrangement of definitions, common notions, and postulates. The last postulate was the Fifth—the Parallel Postulate. This assumption was of a character different from all the preceding ones, and it was looked upon with suspicion and accepted with reluctance immediately after the appearance of Euclid's work. But the introduction by Euclid of his Parallel Postulate was no accident; it was a monument to his insight and skill as a mathematician. The evidence is in the *Elements*. Euclid tried to prove everything he could without this postulate; he even introduced peculiar propositions with but one intent—to prove all he could without the postulate. Finally he had to use the Parallel Postulate.

Much of the history of geometry after the fall of the Alexandrian School is concerned with attempts and failures to prove the Parallel Postulate on the basis of Euclid's other assumptions. These attempts and failures finally led to the discovery of what we now call non-Euclidean geometries, which plainly showed the importance and necessity of Euclid's Parallel Postulate (or some logically equivalent one) for the completion of Euclid's work. But they did more than that; they opened up wide vistas of mathematical progress uninhibited by the doctrine of the infallibility of Euclid.

The first real progress was made in 1733 by the Jesuit monk, Girolama Saccheri, who, however, repudiated his own achievements. He entitled his work "Euclides ab omni naevo vindicatus," or, freely translated, "Euclid freed of every blemish." Saccheri denied an assumption that is logically equivalent to Euclid's Parallel Postulate but retained all the rest of Euclid's assumptions. He developed a logically consistent body of theorems for a geometry which differs from that of Euclid. But he was so convinced of the unthinkability of any geometry other than that of Euclid that, in a final chapter, he lost himself in a morass of philosophical meanderings in the realm of the infinitesimal and rejected all his own correct reasoning. If Saccheri had had a little more imagination and had not been so convinced that there could be no geometry other than that of Euclid, he would have anticipated by a century the discoveries of Gauss, Lobachevski, and Bolyai. Indeed, the Parallel Postulate was not one of the mistakes of Euclid; it was one of Euclid's crowning mathematical achievements.

Let us not say much more about the history of non-Euclidean geometry except to remark that Gauss had developed many ideas about the new geometry by 1800. However, he was a perfectionist and realized that anyone who dealt with the subject was deemed "a bit queer," so he published nothing. But Lobachevski, in 1829, had the courage of his convictions, and, although the imprint of Saccheri's work is plainly visible in his writings, Lobachevski's attitude was different and he gave a complete development of hyperbolic geometry. Today the Russians are very proud indeed of this mathematical hero. About the same time, the work of the younger Bolyai was sent by his father to Gauss, who replied that he had been in possession of much of this material a long time. But all of these writers had still placed too much faith in Euclid, and a more open attitude did not appear until Riemann presented his famous dissertation in 1854. Starting from an entirely different point of view (that of differential geometry), Riemann demonstrated the existence of so-called elliptic geometry, which his predecessors had rejected, and pointed out that others of Euclid's postulates needed a careful scrutiny. The importance of the work of these men does not lie in the discovery of new geometries, but rather in that it caused a crisis in mathematical thinking which led to a critical examination of Euclid's *Elements* and to the discovery of many mistakes made by

Euclid and to methods of correcting these mistakes. Indeed, the work in non-Euclidean geometry led to a careful study of the foundations of all branches of mathematics.

The mistakes of Euclid

Euclid recognized the importance and necessity of starting with appropriate definitions and assumptions. He even went to unnecessary and inadvisable lengths to define every term, although he was acquainted with the works of Aristotle and the latter's statement, "It is not everything that can be proved. You must begin somewhere." Euclid took this into account with respect to his common notions and postulates, but he overlooked the corresponding idea with respect to his definitions. (Unfortunately, this has not yet been recognized by many modern textbook writers.)

Felix Klein wrote in 1908: "The ideal purpose which Euclid had in mind was obviously *the logical derivation of geometric theorems from a set of premises completely laid down in advance.* But Euclid did not, by any means, reach his high goal. Nevertheless, tradition is so strong that Euclid's presentation is widely thought of today as the unexcelled pattern for the foundations of geometry" [2].

It is time to choose. We must either be a follower of Saccheri, and, believing in the infallibility of Euclid, model our texts upon Euclid's *Elements*, or we must follow modern mathematical thought and bring geometry in the schools up to date by achieving Euclid's high ideal—"*the logical derivation of geometric theorems from a set of premises completely laid down in advance.*"

What are some of the mistakes of Euclid? There is not space to bring in a complete bill of particulars, [3] and we must resort to general comments.

1. *Euclid tried to define every term.*

2. *The definitions are not always precise and meaningful.*

It is only necessary to examine Euclid's "definitions" of point, line, and plane to discover this, and to recognize that they play no [illegible] whatever in the logical development of geometry.

3. *The postulates are not always stated precisely.*

The Postulate of Extension is one example. Its uncritical acceptance by the early workers in non-Euclidean geometry led to their rejection of elliptic geometry.

4. *The common notions (or axioms) are not stated in language which is mathematically explicit, and they do not use the terms to which they are applied.*

5. *The postulate system is incomplete.*

The entire omission of any consideration of order, separation, and betweenness accounts for the errors of omission in the *theorem of the exterior angle*, which theorem is fundamental for Euclid's development of the theory of parallel lines—and it is all so unnecessary. A correct, much simpler proof of this theorem is available. Euclid is not entirely blameless, because Eudoxus before him had paved the way for a correct presentation. We might excuse Euclid, however, for historical reasons. But we cannot excuse modern textbook writers who follow Euclid blindly, for not only had Archimedes pointed out the necessity of considering order relations, but geometers have been stressing this point for nearly a hundred years.

The proof of Proposition 1, Book I, is incomplete, because of the lack of a *circle axiom.* This is typical of most of Euclid's construction theory. True, in Propositions 20 and 22 (the *triangle inequality theorem*), Euclid states a necessary condition for the construction of a triangle, but he does not mention the word *sufficient*, and that is what is needed. Anyway, this is a little late for Proposition 1.

6. *Some of the proofs are logically incomplete.*

One must understand that there are no incorrect propositions, merely incomplete (and therefore incorrect) proofs. An outstanding example is Euclid's proof that:

"If two planes have a point in common, they have a line in common." Actually, the proposition is true in 3-space and false in 4-space. The proof is entirely intuitive and without any logical basis.

The outstanding examples of proof based upon intuition rather than logic are those involving the introduction of the *physical* concept of motion. It is true that a mathematical system of postulates and definitions for motion can be made, but this point of view is difficult and complex to develop, and is needless. All superposition proofs contain half a dozen or more implicit assumptions. It is no wonder that Birkhoff and Beatley, in the Preface to their *Basic Geometry*, call such "proofs" *demoralizing*, and this point of view is upheld by any realistic mathematician. But the situation is even worse. Klein says, "The only conceivable purpose for Euclid's Propositions 1, 2, 3 is to avoid the use of physical motion in order to prove Proposition 4, the (*SAS*) *theorem*. But then Euclid did not use them, perhaps recognizing that they were not sufficient for that purpose."

The need for analysis

Euclid's presentation is so formal and ritualistic that it encourages memorizing rather than thinking; it omits all consideration of the analysis of a problem and merely presents a formal synthesis of the proof. Such ideas are directly out of line with those of Archimedes, the greatest of the ancient mathematicians, and with our own modern concepts of learning.

Such a presentation gave rise to the false feeling that elementary geometry had attained perfection, and so delayed progress for a long time. It greatly underestimates the value of analysis in thinking. Making an analysis first is much more in the spirit of scientific development than is the synthesis or polishing of the results. Indeed, after an analysis is made, the attempted synthesis may show the analysis to be incorrect or incomplete, and may point out errors of omission, as in Euclid's proof of the (*SSS*) *theorem* (where sums are used when, in some cases, differences are called for). This points up the fact that Euclid's use of unsigned magnitudes is out of step with even elementary algebra, where signed magnitudes are important.

When the analysis is correct and complete, then the synthesis may suggest further consideration of related ideas. Excellent examples appear in a study by analysis of the *isosceles triangle theorem* and its converse, using different auxiliary constructions. The need for analysis is recognized in part by most of our modern textbook writers because of its pedagogical implications. Some lip service is given to the idea in current texts, if it can be done without interfering with the two-column format. But mere reference to §§ 22, 76, and 103 to cover the method of analysis, leaves the student and the instructor up in the air. Is it any wonder that geometry—the ideal reasoning discipline—became a matter of memory only, and is learned like a catechism?

II. The how:

Do not repeat the mistakes of Euclid

It is no wonder, then, that Klein states in essence, "So many essential difficulties present themselves, precisely near the beginning of the first book, that there can be no talk about the attainment by Euclid of his ideal." Nor is it any wonder that Hilbert, Birkhoff, and others, as well as the Commission on Mathematics, the Illinois Study Group, and the School Mathematics Study Group (sponsored by important mathematics organizations and various foundations) all reached the conclusion that to attain the ideal goal of Euclid *we must have a fresh start.* It will not be Euclid, but it will be geometry that is completely vindicated. For the past sixty years mathematicians have been trying to rewrite Euclid's *Elements* in a form satisfactory for use in the schools, and today success seems near but not

certain. This writing is done in the spirit of Euclid—that is, it involves logical deduction from a relatively small number of explicitly stated assumptions—but an effort is made not to repeat the mistakes.

1. *We must begin formal geometry with undefined terms, stating explicitly what terms are undefined, and we must make precise postulates about these undefined terms.* It is not necessary and is actually undesirable to list *all* postulates at once; it is better to begin with a few and to see what can be done with these few. All this does not mean that a student should start the study of geometry in the formal way. Indeed, much informal geometry should be presented throughout the early grades.

2. *We make definitions for convenience only, and these definitions must be precise and never circumlocutory.* We introduce more postulates and definitions as they are needed. At the end, or in earlier summaries, they can be collected and discussed further if that is desired.

3. *It is not important* what *we assume, if the assumption is inconvenient to prove.* It is essential to recognize every assumption we use and assiduously to avoid making proofs depend upon intuitive preconceived ideas or upon carefully drawn figures. A figure has a very important heuristic value, but the fact that a figure must be used to complete a proof is prima facie evidence that something is wrong.

4. *We should plan the course in geometry so that it can be integrated with other branches of mathematics.* We should bear in mind the connection between algebra and geometry, and look ahead to analytic geometry. We should, however, never lose sight of the fact that there can be no analytic geometry in the sense of Descartes without first having geometry with its theory of congruent triangles, parallel lines, similar triangles, and the theorem of Pythagoras.

The Hilbert or synthetic approach

After the mistakes of Euclid have been pointed out, it is not too difficult to correct them by introducing the needed assumptions. This is precisely what Hilbert did [4]. His idea was to stay as near the form of Euclid as possible and to supply precise postulates which can be made the basis of correct proofs of all the propositions of Euclid's *Elements*. These postulates are concerned with the following ideas: (1) axioms of incidence, (2) axioms of order of points on a line and the extension of these ideas to order relations in a plane by means of some type of transversal axiom (Hilbert used an axiom due to Pasch), (3) axioms concerning and relating congruent segments and congruent angles. These are the essential ideas needed to fill in the gaps left by Euclid. To complete the system, a parallel postulate and a continuity postulate are required.

Euclid did not write for schoolboys, but for the scholars and philosophers of his day. Neither did Hilbert write for schoolboys. The Hilbert approach is too sophisticated for a beginning course in geometry. The preliminary report of the School Mathematics Study Group Committee on Tenth Grade Mathematics states that a tenth-grade course based on Hilbert's postulates would, in its opinion, be so unteachable as to be ridiculous. This is borne out by the initial failure of the Illinois experiments, which did not build enough upon the student's experience. The Illinois program is under modification, and an attempt is being made to give the student more algebraic experience before the study of geometry is begun. The School Mathematics Study Group approach uses many of Hilbert's ideas, but combines them with other very important concepts.

The Birkhoff or arithmetic approach

What then is the solution to the problem? Arithmetize geometry! This is the Birkhoff approach, based upon the student's knowledge of arithmetic and very elementary algebra, and the simple use of a scale and a protractor. It is the basis of the School Mathematics Study Group

approach to geometry. The Study Group is writing both text material and a teacher's manual along this line. The fundamental axioms are the Postulate of Linear Measure and the Postulate of Angular Measure, based upon arithmetical ideas familiar to the student. Congruence is introduced by definition. These are precisely the most important ideas needed to study trigonometry and analytic geometry. They make use of the learner's experience with the scale and protractor and bypass the difficulties (due to the student's age) of continuity. The approach employs strong assumptions, but it does not try to fool anybody. These assumptions supply plenty of exercise in proving easy theorems at the very beginning, and *proof*, not *memory*, is emphasized at the outset. Of course other postulates are made as needed —a Postulate of Separation of the plane by a line, a Parallel Postulate in Playfair's form, and the (SAS) Postulate. A Proportionality Postulate, to develop the theory of similar triangles, can be made, but after much discussion the Group decided to prove the *basic similarity theorem* on the basis of the area concept (Legendre's approach), after presenting the postulates needed to develop the concept of the measure of area. This again enables one to bypass the difficult continuity concept, as well as to give the background for several proofs of the theorem of Pythagoras. The groundwork for the study of circles and spheres, either synthetically or by means of Cartesian co-ordinates in two or three dimensions, has been laid. How far one goes is a matter of the audience.

Let us close with a misquotation: "The good that men do lives after them; their mistakes are often interred with their bones." So may it be with Euclid, so that his glory and ideal shall live forever, with geometry completely vindicated.

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The concept of function[1]

by J. F. Wampler, Nebraska Wesleyan University, Lincoln, Nebraska

It has become evident that most of the current college texts which attempt to integrate the traditional topics of first year mathematics are using as their unifying concept that of function. Because of the great importance of this concept in mathematics and because of the recent emphasis upon it as a unifying principle, I feel that we should be particularly careful in our presentation of this idea to students. I believe that there is a serious need for agreement on the part of mathematics teachers concerning the definition, notation, and language used in working with functions.

Eves and Newsom give a concise account of the development of the function concept from Descartes to the present

[1] Presented to the Nebraska Section of the Mathematical Association of America, April 18, 1959.

day. The early part of this account reads as follows:[2]

> The concept of function . . . has undergone a marked evolution, and every student of mathematics encounters various refinements of this evolution as his studies progress from the elementary courses of high school into the more advanced and sophisticated courses of the graduate college level.
>
> The history of the term "function" furnishes an interesting example of the tendency of mathematicians to generalize and extend their concepts. The word "function" seems to have been introduced by Descartes in 1637, who employed the term simply to mean some positive integral power, x^n, of a variable x. Somewhat later, Leibnitz (1646–1716) employed the term to denote any quantity connected with a curve, such as the coordinates of a point on the curve, the slope of the curve, and so on. Johann Bernoulli (1667–1748) regarded a function as any expression made up of a variable and some constants, and Euler (1707–1783) regarded a function as any equation or formula involving variables and constants. This latter idea is the notion of a function formed by most students of elementary mathematics courses. The Euler concept remained unchanged until Fourier (1768–1830) was led, in his investigations of heat flow, to consider so-called trigonometric series. These series involve a more general type of relationship between variables than had previously been studied, and, in an attempt to furnish a definition of function broad enough to encompass such relationships, Lejeune Dirichlet (1805–1859) arrived at the following formulation: A *variable* is a symbol which represents any one of a set of numbers; if two variables x and y are so related that whenever a value is assigned to x there is automatically assigned, by some rule or correspondence, a value to y, then we say y is a *function* of x. . . .
>
> The student of mathematics usually meets the Dirichlet definition of function in his introductory course in calculus. The definition is a very broad one and does not imply anything regarding the possibility of expressing the relationship between x and y by some kind of analytic expression; it stresses the basic idea of a relationship between two sets of numbers.

I recall that when I first met the Dirichlet definition of function in my introductory course in calculus, I thought about and tried to understand the definition, and I constantly encountered the question, "But what is the function?" Is it the variable y? Is it the rule or correspondence? Or is it somehow both of these or perhaps more? The definition seems to say that if I consider a set of numbers (say the set of nonnegative reals), then for each value of x (a nonnegative real) there is automatically assigned, by some rule (say "the number whose square is equal to x"), a value to y (in this case also a real number). If by function is meant the variable y, then there is some ambiguity in this example because, for $x=4$, y could be either 2 or -2. If by function is meant the rule, then in this example it might be stated as the equation $y^2=x$. This, however, can also be confusing since the equation does not indicate what values may be assigned to x. Is the function then a system consisting of a set of numbers and a rule? To further confuse the concept, according to another frequently used definition of function the above example is not a function at all. Perhaps this illustrates the need for further clarification of this concept.

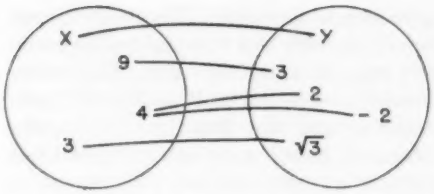


Figure 1

In an attempt to clarify this definition, many authors have simply defined the function to be the rule, or, to be a bit more sophisticated, the mapping of one set, A, into another set, B. The Kansas Summer Writing Group, which wrote *Universal Mathematics, Vol. 1*, published by the Mathematical Association of America, decided upon the latter course—that is, to define a function as a system composed of two sets along with the mapping.

The most recent definitions of function utilize the concept of set and are generalizations of the Dirichlet definition given above. Consider two sets A and B,

[2] Howard Eves and Carroll V. Newsom, *An Introduction to the Foundations and Fundamental Concepts of Mathematics* (New York: Rinehart & Company, Inc., 1957), pp. 250–251. Quoted by permission.

whose elements are not necessarily numbers. The set

$$\{(x, y); x \in A, y \in B\}$$

is called the *cross product* of A and B, and is denoted by $A \times B$. Any subset of $A \times B$ is called a *relation*. For example, consider the set $I \times I$, where I represents the set of integers, and let $x \in I$ and $y \in I$. Then the set

$$R = \{(x, y); x < y\}$$

is a relation. Notice that for any ordered pair of integers, say $(2, 5)$, either $(2, 5) \in R$ or $(2, 5) \notin R$. Instead of writing $(2, 5) \in R$, we usually write $2\ R\ 5$, or since the symbol for "less than" is "$<$", we write $2 < 5$. The set of all first elements of the ordered pairs belonging to R is called the *domain* of the relation, and the set of all second elements of the ordered pairs belonging to R is called the *range* of the relation. Notice that in a relation there may be two ordered pairs with the same first element. In our example, $2\ R\ 5$ and also $2\ R\ 4$, as well as an infinite number of other ordered pairs with 2 as the first element.

If a relation R is such that no two ordered pairs belonging to R have the same first elements, that is, if $(x, y) \in R$ and $(x, z) \in R$ implies that $y = z$, then the relation is called a *function*. Notice that the function is a set of ordered pairs whose first elements belong to the domain and whose second elements belong to the range. If the name of the function is f, then $f(x)$ is used to denote the second element of the ordered pair belonging to f which has the first element x; $f(x)$ is called the *value* of the function f at x. For example, the trigonometric function called *sine* would consist of a set of ordered pairs of real numbers whose domain is the set of real numbers and whose range would be the set of real numbers in the interval $[-1, 1]$. Then $\sin(\pi/2)$ would denote the second element of the ordered pair belonging to sine whose first element is $\pi/2$. It of course follows then that $\sin(\pi/2) = 1$. Should this definition be adopted as standard for high schools and colleges, it would mean a change in notation and a much more careful use of our language.

A final example will now be given to illustrate how the use of this latter definition of function might make generalization more meaningful for certain other concepts in mathematics. Consider a function f: (x, y) whose domain A and range B are subsets of the set of reals. Let a be an element of A, and b be an element of B. We say b is the *limit of the function f at a* if for every real number $\epsilon > 0$ there exists a real number δ such that if

$$0 < |x - a| < \delta,$$

then

$$|f(x) - b| < \epsilon.$$

That is, if x belongs to the set of real numbers, excluding a, on the interval $(a - \delta, a + \delta)$, then the value $f(x)$ belongs to the set of real numbers on the interval $(b - \epsilon, b + \epsilon)$.

To generalize this definition of limit of a function, suppose the function f: (x, y) has domain A and range B, each of which is a subset of a Hausdorff space H. Let a be an element of the closure of A in H, and let b be an element of B. Then b is the limit of the function f at a if for every neighborhood N_b of b there exists a neighborhood N_a of a such that if

$$x \in N_a \cap A - \{a\}$$

then $f(x) \in N_b$.

In summary, the definition of a function as a set of ordered pairs has several advantages over the Dirichlet definition. The former is easier to teach and easier to understand because it is more precise and less ambiguous. This definition facilitates work with graphing functions since each element of a function is merely interpreted as coordinates of a point in the plane. Finally, as illustrated, this definition of function facilitates generalization of definitions and theorems concerning the function concept.

Reviews and evaluations

Edited by Kenneth B. Henderson, University of Illinois, Urbana, Illinois

BOOKS

Analytic Geometry and an Introduction to Calculus, A. Clyde Schock and Bernard S. Warshaw (Englewood Cliffs, N.J.: Prentice-Hall, Inc., 1960). Cloth, 165 pp., $3.96.

According to the authors, this book is ". . . written to help bridge the gap between the mathematics usually taught in the senior year of high school and this mathematics of the freshman year of college." The book could be used as a text in a survey course in analytic geometry and the introduction to calculus, but the treatment is by no means rigorous or thorough. This might be the answer for someone looking for a text for a one-semester course in the fourth year of high school. It would not be sufficient for preparing students for advanced placement.

The analytic geometry is approached in the traditional way. Without giving a definition of a variable, the authors define a function as a relationship between two variables such that for each value of one variable, there corresponds one value of the other. The graph of a function is presented as a "smooth curve" connecting a number of points found by substituting values for the independent variable, and getting the corresponding values for the dependent variable. At several places in the text the curve is described as being generated by a moving point. Only in the case of the first degree equation is there a proof that the equation can be represented graphically by a straight line. The conics are defined as loci, and the standard forms are derived. There is, however, no mention made of the fact that every number pair which satisfies the equation represents a point on the locus. There is also no mention made of the degenerate conics, or of equations which have no graphs. It is also suggested that all curves be sketched neatly without the use of graph paper.

The chapter on polar co-ordinates develops only the simple equation for the line through the pole, and the equations for the circles with center at the pole or on the polar axis. A few of the other loci in polar form are given in the list of exercises. There is no material on parametric equations nor on the trigonometric functions. There is only a brief section on the exponential and logarithmic functions, and the hyperbolic functions are omitted.

In the introduction to the calculus, the concept of limit is arrived at intuitively, and then presented as a general definition. The derivations of the formulas for the differentiation of the algebraic functions are quite clear and good. Again, there is no material on the trigonometric or exponential functions. Integration is presented inductively as the inverse process of differentiation. There is a chapter on the applications of the derivative with a section on maxima and minima, and one on related rates. There is, however, only a limited amount of material on the applications of integral calculus.

There is a sufficient number of problems in each section, but they are not varied. Most of the problems closely follow the examples worked out in the text. There are very few challenging questions for the better student. All of the formulas developed are collected in an appendix, and the answers to most of the odd-numbered problems. are given.—*Leroy Sachs, Clayton High School, Clayton, Missouri.*

Elementary Statistics, Sidney F. Mack (New York: Henry Holt and Company, Inc., 1960). Cloth, ix+198 pp., $4.50.

Society's demand for still more technology, and its increasing faith in the need for applying the scientific method to more fields of endeavor, has created a need for educational institutions to provide more training in mathematics and statistics. *Elementary Statistics* is a concise textbook written for an elective statistics course designed to give mathematics credit to college freshman or sophomore students in nontechnical curricula. The book could be divided basically into four parts: (1) the development of elementary probability; (2) probability distributions and statistical estimation; (3) the rationale and process of testing hypotheses; and (4) the development and interpretation of linear correlation and regression.

Where appropriate, the brief and cogent presentation is enhanced by excellent figures and diagrams which clearly portray the concepts under discussion. This is a rare combination found in few statistics textbooks. The concise and "fast-moving" approach is preserved by placing the treatments of complicated derivations of important concepts in the appendixes. To assure and emphasize the position of these concepts in the development of the discipline, general statements are made about them, each of which is labeled a "theorem." These statements are supported by illustrative

examples and instructive problems which further clarify the concepts involved. Although the book is brief, brevity has not ruled to the extent that it impairs the presentation. A brief treatment of any subject necessitates the sacrifice of certain elements and refinements. The sacrifice here was in the depth of examination of the topics covered. This sacrifice, however, in no way causes any vagueness or lack of clarity. The general theme stands out clearly, and its continuity is preserved throughout.

Some readers might feel that author Mack is guilty of mathematical "obscenities," because he uses freely such nasty words as "transposing" and "canceling." The forthright style of the author, however, lends an informality to the discussion that in all probability makes the subject more accessible to the student with a limited mathematics background. Although the book has been written for college freshman or sophomore students with "a minimum of one semester of elementary college algebra or a good grounding in high-school algebra," there is no reason why it could not be used as a textbook for qualified high school students in a senior year "honors" statistics course.

Since a background in algebra is required of students who use the book, most of the material covered in the "Mathematics Refresher" chapter could have been omitted. By the same token, less coverage could have been accorded certain details of elementary probability, with more emphasis on testing hypotheses. This would have resulted in a better "balance" between probability theory per se and its applied role in the field of statistics. Except for these minor difficulties, and an unfortunate "multiple-usage" of the letter "*l*" which causes some confusion in symbolism, the book is well written, and probably will prove to be one of the first successful efforts in the "short book" approach to the broad field of elementary statistics.—*Joseph E. Hill, Wayne State University, Detroit, Michigan.*

Elements of Algebra, 3rd ed., Howard Levi (New York: Chelsea Publishing Company, 1960). Cloth, xi+161 pp., $3.25.

This book is a refreshing change from the usual run of mathematics textbooks. It makes every effort to preserve the logical elegance of the subject and to give the reader a glimpse into the heights to which mathematics can take him. But, in the opinion of the reviewer, it seems doubtful whether the author is likely to achieve his aim with anyone but the "initiated"—i.e., the mathematically gifted or sophisticated.

The author intends his book as the text for a course serving two purposes. The first one—that of providing "the first of a sequence of mathematical courses leading to advanced work in mathematics and related subjects"—the reviewer agrees is likely to be served by the book. It should also be a good screening device for separating the sheep from the goats. It is not likely to fail to fire with enthusiasm the mathematically gifted student, and to give him a taste for the joys in store for him. On the other hand, the student not gifted with either a love for the logical or the ability to comprehend the intricacies of the subject is very likely to be scared away from the field, although he will do so with a healthy respect for what he is leaving behind.

There seems no reason to doubt that the second purpose—that of ensuring that "the liberal arts student" will be "exposed to genuine mathematical problems"—will be satisfied by the book; but the question remains whether the effects of this exposure will always be good. The reviewer fears that the diet might prove too rich for the general run of students.

While on the whole the development of topics is truly masterly, there are a few weaknesses to which it is necessary to call attention. The author fails to make clear enough the distinction between symbols and the objects the symbols designate. This is especially to be deplored since the distinction becomes a necessary one in the development of certain concepts. For instance, a variable is defined as a symbol. It is said to play "the same role as the blank space." It is said to refer to a set which is called its domain—i.e., it serves to designate the members of a set. In other words, according to the author a variable is a *symbol* which *stands for* any one of the members of a specified set. It does not then seem correct to the reviewer to speak of replacing a variable "by a member of its domain." The variable may be replaced by a *symbol* which designates a member of its domain, but it cannot be replaced by the thing itself. While it may be argued that this is too trivial a point, the reviewer would like to point out that in a book which purports to be a representation of the finer points of mathematics such a confusion could defeat the purpose for which the book was written.

Another place where the failure to make this distinction clearly is likely to interfere with the clear formation of a concept is in the section dealing with the definition of a cardinal number. The author defines "the *cardinal number* of a finite set to be *the name* (italics not in the original) of the standard set. . . . " But in the ensuing section it is clear that a cardinal number is not a name but the thing designated by the name—e.g., " . . . it turns out that 12 and 7+5 designate the same cardinal number." This confusion persists until one almost wonders whether the author meant to call both the symbol and the thing designated by the symbol "a cardinal number." On page 35 he says, "If each variable . . . is replaced by a cardinal number, then the combination of symbols thus obtained designates a cardinal number."

There is another topic that could have been treated more precisely. In talking about statements the author often talks of "proving statements" by which he means proving that the

statements are true. This could very well lead to confusion, especially as some theorems are proved by proving certain statements to be false. This implies that statements could be proved either true or false, so that the phrase "proving a statement" is ambiguous in its reference.

The next criticism bears on the book as a textbook. It is too concise and concentrated to be a good textbook. Very many of the discussions are extremely abstract and rather unclear, and often no concrete examples are given to illustrate the point. This would not be too much of a drawback in the hands of a skillful teacher who is also a competent mathematician able to supply the necessary explanations and examples. But should a textbook presuppose such fortunate circumstances? The fact that no answers are supplied for the exercises at the end of each chapter also points to the necessity for a sophisticated teacher.

Finally, the reviewer found the numbering of theorems unsatisfactory. The theorems are numbered afresh in each chapter, making it necessary for anyone referring to them to quote the chapter as well as the number of the theorem. This is an inconvenience which could have been easily avoided by numbering the theorems in a continuing sequence without reference to chapters.

On the whole, although the book is intellectually stimulating, in its present form it is not likely to be very useful as a textbook except in the hands of a skillful and sophisticated teacher. There is much to commend the book, and the reviewer feels that all serious students of mathematics would benefit by reading it. It provides not only a valuable intellectual experience but also a pleasant excursion into the heights of abstraction.—*Nageswari Rajaratnam, University of Illinois, Urbana, Illinois.*

General Mathematics, Lucien B. Kinney, Vincent Ruble, and M. Russell Blythe (New York: Henry Holt and Company, 1960). Cloth, x+595 pp., $4.20.

It seems reasonable to evaluate a textbook in terms of how well it accomplishes its stated purposes. In their *Teacher's Manual* the authors list the following three purposes for their book:

"(1) To provide for activities in the classroom and outside that will make arithmetic interesting and meaningful. (2) To introduce many applications of arithmetic at the level of the student's interest so that he can see the value of arithmetic and its relationship to and importance in everyday affairs of living. (3) To introduce experience in algebra and geometry that will be useful and meaningful to the student, and also to test out the ability of the student to handle these subjects."

In this reviewer's opinion, *General Mathematics* only partially fulfills these purposes.

The "interest" portion of the first objective is difficult to assess, but with the exception of optional problems for the better student (which are generally several cuts above the average), the usual topics applied in the usual situations appear throughout the book. The photography, art work, and use of color are certainly assets. In judging how the book promotes "meaningful" learning of arithmetic, one must stipulate how the term "meaningful" is being used. If by "meaningful" the authors mean a clear exposition of prescriptions for performing arithmetic processes plus examples and practice, then (with some exceptions) they have succeeded. Moreover, their emphasis on the steps of problem-solving is handled in an exemplary manner. Practice in each step of the process and frequent review of the procedure may be found throughout the book. On the other hand, if "meaningful" refers to understanding or insight into the structure of arithmetic beyond the "how-to" level, then this book is unsatisfactory. Let me quote from the development of division of fractions (pp. 144–145) to illustrate:

"To divide 5 by 5/8, write 5 as an improper fraction, with 8 as the denominator. 40/8÷5/8. Then divide numerator by numerator (40÷5=8) and denominator by denominator (8÷8=1). You have: 40/8÷5/8=8/1=8. A quicker way is to *put* [italics mine] 5 in the denominator and 8 in the numerator. Then we can use cancellation, thus:

$$\frac{1}{5}/\frac{1}{1}\times\frac{8}{\cancel{5}}=\frac{8}{1}=8.$$

Here is a simple rule for dividing by a fraction or mixed number:

Rule: 1. Change any mixed number to an improper fraction.
2. Invert the divisor and multiply."

Loose language is found in a number of places in the book. For example: "To reduce a common fraction to lowest terms, divide both the numerator and denominator by the same number—the largest that will go into both terms" (p. 167). Thus, one might logically defend the following: to reduce 5/6 to lowest terms, divide numerator and denominator by 5 since 5 goes into both 5 and 6. The Glossary affords more examples of loose language. These may be found on its first page:

"Algebra: expressing number ideas with letters or symbols.

Angle: the figure formed by two intersecting lines.

Approximate number: a number that represents a nearly, but not precisely, exact measurement.

Base: the line or surface on which a plane figure or solid rests."

The book's second purpose, "to introduce many applications of arithmetic . . . ," is successfully accomplished as long as we take

"introduce" to mean something other than, "for the first time." Authors of ninth-grade general mathematics books are hard pressed to find suitable arithmetic applications which have not already been exploited by the upper elementary and seventh and eighth grade texts.

The chapter devoted to algebra deviates from the prescriptive "how-to" exposition so common elsewhere in the book. The solution of simple linear equations is presented in an intuitive manner by using the analogy of the balance. The work in geometry primarily deals with a variety of constructions presented in a prescriptive manner; the triangle; and areas and volumes of various figures.

A maintenance and remedial program is included, with periodic tests related by references to practice work in the final chapter. The *Teacher's Manual* contains suggested classroom objectives and procedures for each section, supplementary work, suggested assignments, and answers to all excercises and questions.

By some standards *General Mathematics* might be an acceptable textbook, but for this reviewer its good features cannot balance out its mathematical and pedagogical deficiencies.—*Robert B. Kane, Purdue University, Lafayette, Indiana.*

MISCELLANEOUS

Praxis der Mathematick: Monatshefte der reinen und der angewandten Mathematick im Unterricht ("Practice of mathematics: Monthly of pure and of applied mathematics in instruction") (Cologne, West Germany: Aulis-Verlag, Teubner and Company KG.). 18DM. per year, plus postage.

This new German mathematical monthly, of which the first issue appeared in April, 1959, is devoted to the problems of teaching mathematics at what corresponds roughly to high school through junior college level in this country. It contains articles, a problem section, book reviews, and news and notices.

The nature of the articles, many of which resemble the "Classroom Notes" of the *American Mathematical Monthly*, is suggested by the following selected list of titles:

On the treatment of number theory in the upper grades
Affinity and central collineation in descriptive geometry
On the inverse relationship of the e-function and the logarithm
Nomogram for determination of gas volumes
The introduction of the root in instruction
Rocket flight as a subject for upper class instruction
The numerical fitting of surfaces
Aristotle and Eudoxus
New results on Babylonian arithmetic, algebra, and geometry (a series and an interesting one)
Stirling numbers and $e^{\ln x} = x = \ln (e^x)$
Linear programming
On the status of mathematical instruction in France
Biological growth functions
A method of treating special relativity

These articles are written in elementary, classical, expository style. They reflect a greater interest in history and a greater concern with applications than is common in American journals. The problems are often of standard textbook type. Contrasting all this with the concern of the corresponding American journals with changing curricula and elementary research, I believe that most of our readers would consider *Praxis* a little behind the times.—*Franz Hohn, University of Illinois, Champaign, Illinois.*

A Survey of Basic Mathematics, H. G. Apostle (Boston: Little, Brown and Company, 1960). Cloth, xv+464 pp. $6.00.

This text is divided into five sections: Arithmetic, Algebra, Geometry, Applications (Probability, Statistics, Logarithms, Trigonometry, Vectors), and Introducing Higher Mathematics (Analytical Geometry, Calculus). There are also an Introduction (Logic) and brief Appendices. The Preface begins thus: "This book has been prepared primarily to meet the mathematical needs of liberal arts students who do not plan to continue the study of mathematics. Little mathematical knowledge beyond the elementary arithmetical operations is presupposed on the part of the student." Any consideration of the text must take this into account.

The appeal to the nonmathematical student is apparent throughout the text. Each topic is presented in rather elementary terms, and in conveniently simple situations. For example, analytical geometry occupies 29 pages, of which only 8 are devoted to conic sections. Each of the conic sections is described verbally and analytically, centered at the origin. There are no troublesome complications for the student to worry (or think) about. Calculus is presented in 36 pages, 28 for differential calculus and 8 for integral calculus. Only polynomial functions are used, and none of higher degree than three.

The text is skillfully written and easy to read. The mathematical ideas, though severely limited in scope, are accurately presented, generally speaking. The essential question is whether or not this is the way in which mathematics should be presented, even to the nonmathematical student. Rigor has been replaced by an appeal to intuition, and the word "evident" is used frequently. Granted that this may at times be desirable, it would still seem important for the nonmathematical student to know the place of rigorous proof in the development

of mathematics. And good proofs require good definitions, which the text often avoids at the expense of excess verbiage. For instance, more than a page is devoted to the meaning of "inverse operations," but still with limited results. One good definition would have sufficed, such as "If f and g are functions which are so related that $(f \circ g)(x) = x$ for every element x in the domain of g and $(g \circ f)(y) = y$ for every element y in the domain of f, then f and g are said to be inverses of each other."[1]

The brevity with which each topic is presented precludes a thorough treatment. The surface is skimmed, rather than the depths probed. Even so, the main topics in arithmetic, algebra, and geometry are adroitly handled, but with a few exceptions. In all of the 92 pages devoted to algebra, there is no mention of inequalities. One would expect to find this topic, with its applications to linear programming, mentioned in a survey course. The treatment of trigonometric functions seems limited, using the definitions in terms of the ratios of the sides of a right triangle. The periodic function approach is more interesting, and much more powerful. Why deny it to liberal arts students? The logarithm is presented without any mention of its domain and range, with no emphasis on its inverse relation to the exponential function, and with no idea of what its graph shows. This reviewer would have preferred a thorough treatment of these and other topics, at the expense of the introductions to analytic geometry and calculus.

It is hard to conceive of this text's fitting into the emerging secondary school curriculum. Its lack of thoroughness, and a corresponding lack of appeal to the student's intellectual curiosity, make it unsuitable. It is conceivable, however, that in certain liberal arts colleges or junior colleges a survey course in mathematics (perhaps roughly comparable to a music appreciation course at an engineering college) is desirable for those students who have forgotten what little mathematics they once knew. For such students, this text would be quite appropriate.—*John K. Moulton, Brookline High School, Brookline, Massachusetts.*

[1] "Elementary Functions," School Mathematics Study Group (New Haven: Yale University, 1960), p. 275.

Much ado about nothing

I am Zero.

Elementary school children and other uneducated people sometimes call me *nothing.*

But, if I must say so myself, I am really *something!*

True, I arrived on the scene rather late in the day. Possibly my so-called *nothingness* made me elusive to grasp.

But I was there all the time, a shadow lurking in the background, laughing to myself at man's resultant ineptitude, because he could not locate me.

Enfin came *Zero!*

People for a long time continued to argue whether I should be allowed into the select circle of numbers or not. But inevitably, they capitulated to the force of logic.

Today, I find myself at the center of the numerical universe. On one side are all the positive real numbers extending to infinity; on the other, there are the negatives; and in all directions, the numbers of the complex domain.

All roads lead to *Zero.* I am the general point of reference in all co-ordinate systems: for a line, it is *Zero;* for a plane, it is *Zero, Zero;* for three-dimensional space, I became *Zero, Zero, Zero;* for four-dimensions, *Zero, Zero, Zero, Zero;* and so on to infinity. Truly, my personality has multiple facets without limit.

Then there is the lowly infinitesimal—a transient being of no particular value which approaches me as a limit. Approaches, I say—through many stages and degrees with something of a sense of awe and confusion.

And who makes all the difference, whether in the field of science, industry, or finance? Just write six of me after any numeral and note the change in significance. Between the numeral for a lowly one and a magnificent million, the difference is six *Zeros.* In the other direction, six *Zeros* behind a decimal point will change a one to one one-millionth. One with a mere hundred *Zeros* after it is called a *Googol,* a figure that staggers the human imagination. But all people have done is place a hundred so-called *Nothings* after one. Oh! the power of Me!

I am Zero!!

And who is master of the situation? The *mathematician,* in his conceited thoughtlessness, believes he has created me. But I was there all the while, only he was too stupid to discover me. Today, he finds me indispensable. But on the other hand, he is bedeviled by me to the point that he has to outlaw division by *Zero!*

O victorious analysis, whither art thou?—*Brother U. Alfred, F.S.C., St. Mary's College, California.*

Edited by Joseph N. Payne, University of Michigan, Ann Arbor, Michigan, and William C. Lowry, University of Virginia, Charlottesville, Virginia

Pet peeves and pitfalls

by W. Rodman Snelling, Morgan Park Academy, Chicago, Illinois

Some time ago a popular national magazine offered a regular feature column which developed a theme that can be paraphrased, "They ought to. . . ." In that column readers noted their pet peeves and dared others to carry through plans to alleviate the situations described. In the same vein, I intend to point out my pet peeves in the teaching of mathematics with the hope that others may avoid the pitfalls that these small problems create in the classroom.

They ought to outlaw the use of the term "transposition" and the sense of the operation which it describes. Students are all too prone to buy a technique without carefully justifying its use. Thus, most students arrive on the secondary school scene firmly embracing the concept of transposition. To them it refers to the *movement* of a term from one side of an equation to the other and involves a change of sign as it passes to the new side. Unfortunately, the dominant idea seems to be one of movement and even the best of students commonly err and fail to alter the sign. I insist that pupils use the phrase "add (or subtract) the term to (or from) each side of the equation." I find that continual stress on this small point results in far fewer "careless" errors.

They ought to rename improper fractions. Somewhere within their elementary education American youths have been carefully instructed in the secrets of transforming an improper fraction into a mixed number. However, the mere term "improper" seems to have conveyed to them the necessity of avoiding improper fractions completely. Whenever they are confronted with such a fraction they want to change it immediately—after all, it is *improper.* They are most reluctant to accept arguments presented by a teacher in favor of retaining the improper fraction form because it lends itself readily to algebraic operations and simplified notation. I have found that a few moments in class devoted to the selection of a new name for this type of fraction pays handsome dividends. Everyone gets into the act, and some name such as "top-heavy" is usually adopted. After this, the students are much more willing to accept the fact that there is nothing improper about an improper fraction except the term "improper" itself.

They ought to justify the rationalization of fractions. Most texts I have encountered make no attempt to justify the need for removing radicals from denominators of fractions. Quite rightfully, most intelligent students will question why the procedure is desirable. Merely to say that it will simplify the fraction and leave it in a form that will combine readily with other quantities is insufficient. Certainly, to the student

$$\frac{1}{\sqrt{3}+\sqrt{2}}$$

looks little different from

$$\frac{\sqrt{3}-\sqrt{2}}{1}.$$

Motivation for the study of rationalization can, however, be quickly created by examining

$$\frac{1}{\sqrt{2}}.$$

Upon multiplication by

$$\frac{\sqrt{2}}{\sqrt{2}},$$

the fraction becomes

$$\frac{\sqrt{2}}{2};$$

and then, accepting 1.414 as an approximate value for $\sqrt{2}$, the students can be challenged to obtain an approximation for

$$\frac{1}{\sqrt{2}}$$

while the teacher obtains an approximation for

$$\frac{\sqrt{2}}{2}.$$

More important mathematically is the fact that the students are using the idea that a set of numbers of a certain form is closed under the operation of division; that is, that they can always divide any two numbers in the set, with the usual exception of division by zero, and obtain a quotient of the same form.

They ought to be more explicit when discussing factors of algebraic expressions. Teachers commonly consider an expression such as x^2-2 to be unfactorable. It can be awkward if a student later notes that the product $(x+\sqrt{2})(x-\sqrt{2})$ is equal to x^2-2. When the factoring of the difference of two squares is first considered, an example such as the one above should be examined and factored; and the teacher and the textbook should point out that to say it cannot be factored means it cannot be factored with rational coefficients. Similarly, it should be pointed out that a polynomial such as x^2+3x+1 can be factored, but not with rational coefficients for the terms in the factors.

They ought to avoid the careless use of the terms "imaginary" and "complex." Many published materials in algebra confuse students by using these two terms interchangeably. The general complex number, $a+bi$, is the superset that includes the set of all reals and the set of all imaginaries. Thus, it would be correct to refer to all imaginary numbers, for example the number $3i$, as complex numbers (the real component is zero). But many authors mistakenly refer to complex numbers as imaginary numbers. Complex numbers merely contain an imaginary component. The real number, 4, is a complex number (the imaginary component is zero), and some students are quick to point out that therefore not all complex numbers are imaginary numbers.

They ought to encourage students of geometry to express the arguments to their proofs in paragraph form. The traditional outline method, involving a divided page with statements on one side and reasons on the other, may be effectively used to demonstrate the essential components of a proof at the beginning of the year. It does not, however, encourage a student to transfer to other areas the logical presentation of an argument so well exemplified by mathematical proof. In an English course he is required to express himself in sentences and paragraphs even when he may be trying to lead a reader to a particular conclusion. I prefer to encourage students to adopt a paragraph form of proof as early in the year as feasible. As an example of the form of proof I prefer, consider the following problem.

GIVEN: $\triangle ABC$ with $\measuredangle BAC = \measuredangle ABC$: P and Q are on AC and BC, respectively; $AP = BQ$; and AQ and BP intersect at R. PROVE: $\triangle ABR$ is isosceles. PROOF: Since it is given that $\measuredangle BAC$ is equal to $\measuredangle ABC$, it follows that $\measuredangle PAB$ equals $\measuredangle QBA$. Furthermore, AB is identically equal to BA, and AP is given equal to BQ. Hence, by S.A.S., triangle APB is congruent to triangle BQA. Angle PBA, therefore, equals $\measuredangle QAB$; and $\measuredangle RBA$ equals $\measuredangle RAB$. Triangle ARB is isosceles, since its base angles are equal.

In using the paragraph proof I try to stress the desirability of concise expression and concern for the ambiguity prevalent in all language. This provides ample opportunity for short discussions on semantics and the beauty of mathematics as a language relatively free from semantic difficulties.

Finally, it is the writer's opinion, although there are those who disagree, that *they—the teachers—ought to require written homework of every mathematics student every day, and it ought to be corrected and returned immediately.* Mathematics cannot be fathomed by high school students unless they actually *do* mathematics. It is, therefore, necessary that students receive extensive experience in mathematical problems designed to extend and broaden their understanding of topics considered in class. It is indeed unfortunate that many school administrators and school boards hold to a belief that mathematics can be taught meaningfully in classes containing more than 25 students. Under such circumstances, and with the normal load of classes, it is manifestly impossible to assign homework and conscientiously correct it; but we should strive toward this goal and adopt whatever means we can to achieve partial fulfillment under existing circumstances.

Recent research has shown that learning is most efficient when correction of errors is immediate. To me, this points up a serious fault committed by many teachers; assigned homework is collected the next day, examined that evening (if then), and returned to the student the following day. Thus the corrected work is returned at least thirty-six hours after it was done by the student. By that time the student has little interest in the teacher's corrections since the class work has proceeded to more advanced topics, and the student's one opportunity for daily, close contact with his instructor has been lost. I do not have any quick solution to this dilemma—the need for daily work to be carefully examined and corrected by the teacher and the problem of insufficient time for it to be corrected and returned. However, see how close you can come to the ideal.

I trust that someone may benefit from reading these pet peeves that have influenced my teaching. At least the expression of them on paper gives relief to the inner self. I encourage all of you who are beginners to make note of your own pet peeves as they develop and to devise ways and means to deal with the problems which have made you aware of your pet peeves.

What's new?

BOOKLETS

Analysis of Research in Teaching of Mathematics, 1957 and 1958, Kenneth E. Brown and John J. Kinsella. U. S. Department of Health, Education and Welfare, Office of Education. Bulletin 1960, No. 8. For sale by the Superintendent of Documents, U. S. Government Printing Office, Washington 25, D.C., 25¢.

Five Little Stories, William W. Strader. Washington, D.C.: National Council of Teachers of Mathematics, 1960. Paper, 16 pp., 50¢.

FILMS

Formulas in Mathematics. Chicago: International Film Bureau, Inc., 1960. 10 minutes, color. $110.

Language of Algebra. Chicago: International Film Bureau, Inc., 1960. 16 minutes, color. $185.

NCTM

THE NATIONAL COUNCIL OF TEACHERS OF MATHEMATICS

Report of the Membership Committee

The Membership Committee of the National Council of Teachers of Mathematics is happy to submit to you its Annual Report and to congratulate you for your outstanding membership achievement as evidenced by the facts we are presenting to you. We are confident you will find these facts most gratifying—surprisingly so, since few of us even hoped for the generous growth which has been accomplished during the past year.

It is quite likely that you have checked totals before referring to this message, so I need only to compare the present total of 31,194 with the goals we set for ourselves during the year.

In our Annual Report at Dallas, Texas, in April 1959, we noted the rapidity of current growth and predicted an easy attainment of the 25,000 goal by the next Annual Meeting in April 1960. Later on—in the October 1959 issue of THE MATHEMATICS TEACHER—we changed that prediction to a still more ambitious goal of 28,000 to be reached in that span of time. We very easily surpassed that goal.

The final membership count for the year—made on May 1, 1960—shows a total of 31,194 members and subscribers, or 3,194 more than the larger goal of 28,000 which we had set for ourselves.

You will be interested to know that *we are now the second largest department affiliated with the NEA. A year from now we should be the largest.*

We must concede that the trends of the times have been exceptionally helpful in bringing about this noteworthy growth. The remarkably fine improvements and generous expansion of services provided by the officers and board members of the Council have been strong added factors in attaining this growth.

A very great deal of the strength of the Council is due the efficient and diligent work of the state representatives supported by the officers and members of the Affiliated Groups. The many, many committee members of the Council working to further NCTM services are making an outstanding contribution to the Council.

Your keen enthusiasm about the growth of the Council and the expansion of its services, and *your direct assistance as individuals* in the realization of this expansion have been major contributing factors toward the growth and increased strength of this great organization. A continuance of this fine support is vital to the success of the Council in its services to mathematics education.

Are we fast approaching the limit of National Council membership? Or is there still considerable room for growth? Statistical data from the U. S. Office of Education would seem to indicate that *our current totals*—excellent as they are—*are less than 50% of potential membership.* Is it reasonable to believe that this potential membership will continue to increase as school enrollments become greater and greater?

Let us consider the National Council membership situation in our own local-

ities. How many of our teachers of mathematics are National Council members? What per cent of the members of local and state Councils of Mathematics are also members of the National Council? May I assume an agreement on your part to great possibilities for continued membership growth?

If this assumption is valid, is it not quite evident that current goals should again be raised if incentive for growth is to remain strong? What advance in goals is reasonable? What procedure should be followed in working toward these goals?

May we suggest a general procedure such as we have found effective in the past —that of raising total goals for individuals and subscribers—doubtless placing greater emphasis in the future on individual memberships. In this manner, it is very probable that we can raise *individual memberships* well above 28,000 during the next year or so.

The present total of individuals is 22,365. Last year's records show 16,312 individual members. This indicates an increase of 6,053 individuals during the current year. Surely, we should at least equal that growth during the coming year. We would accordingly hope for about 28,500 individual members by May 1, 1961.

Currently, individual memberships comprise about 72% of grand totals. Based on the estimated 28,500 total of individuals suggested in the above paragraph, our grand total by May 1, 1961 should be approximately 39,500. This presupposes an over-all growth next year equal to the increase experienced during the past twelve months.

Let us look at the problem in another way. Let us examine the year-by-year growth in grand totals during the past six years:

Increase from May 1954 to May 1955....2723
Increase from May 1955 to May 1956....1103
Increase from May 1956 to May 1957....2354
Increase from May 1957 to May 1958....3316
Increase from May 1958 to May 1959....4293
Increase from May 1959 to May 1960....7404

After the first year's "spurt" in membership growth, records—until this past year—show that each year's increase in members was approximately 1000 more than the increase for the preceding year. May we not anticipate approximately 8500 more members for the current year just beginning?

If this trend continues, is a new goal of 40,000 members and subscribers too ambitious? Perhaps we should give ourselves two years to meet this goal. This achievement should result in approximately 29,000 individual members.

A report similar to that presented above was submitted to the officers and board members at the Buffalo meeting. The report was accepted with enthusiasm and keen appreciation by these officials; the recommendations were considered very reasonable and quite possible to be realized in the near future.

A similar report with recommendations for increased goals was also submitted to the delegates at the Eleventh Delegate Assembly. Here more satisfaction than surprise was in evidence over the reported achievements. These achievements were generally considered merely as a good beginning for still greater membership progress.

It was universally agreed that prompt and continued action was urgent; that now was the time to make maximum gains while interest in the improvement of mathematics education was at a high peak.

Action was taken to accept the recommendation for a 40,000 membership goal. A time limit of one year was set for the accomplishment of this new objective.

This resolution was also passed by the state representatives at their Annual Meeting.

The following pages present a detailed analysis of National Council membership achievement by state and territory. Please study this for the over-all picture it presents and for the specific data which concern your more immediate interests.

Analysis of Membership Growth, Members and Subscribers, May 1959 to May 1960

	May 1959	May 1960		Goal and Per Cent of Goal Reached	
		Individuals	Totals	25,000 Goal	Per Cent
Alabama	214	221	316	282	112%
Alaska	17	12	19	20	95%
Arizona	278	265	339	100	339%
Arkansas	197	155	227	285	80%
California	1,765	1,493	2,455	1,465	167%
Colorado	294	319	402	300	134%
Connecticut	386	357	503	380	132%
Delaware	87	83	111	117	95%
District of Columbia	237	208	239	307	78%
Florida	624	547	740	585	126%
Georgia	248	272	527	310	170%
Hawaii	65	99	137	76	180%
Idaho	38	28	39	30	130%
Illinois	1,810	1,622	2,157	1,980	109%
Indiana	623	753	916	899	102%
Iowa	391	390	522	492	106%
Kansas	411	425	513	517	99%
Kentucky	188	185	240	205	117%
Louisiana	398	351	518	460	113%
Maine	101	111	137	115	119%
Maryland	390	383	529	475	111%
Massachusetts	648	640	867	732	118%
Michigan	1,200	1,248	1,852	1,000	185%
Minnesota	622	690	889	667	133%
Mississippi	175	159	209	205	102%
Missouri	438	423	549	550	99%
Montana	117	120	148	100	148%
Nebraska	188	208	268	275	97%
Nevada	42	29	44	22	200%
New Hampshire	106	102	128	107	120%
New Jersey	961	835	1,152	935	123%
New Mexico	114	112	166	147	113%
New York	2,184	2,128	2,893	1,967	147%
North Carolina	274	285	459	425	108%
North Dakota	68	101	127	72	176%
Ohio	1,099	1,091	1,345	1,232	109%
Oklahoma	375	327	472	395	120%
Oregon	306	298	437	222	197%
Pennsylvania	1,486	1,437	1,896	1,532	124%
Rhode Island	101	109	144	117	122%
South Carolina	160	144	207	227	91%
South Dakota	79	61	90	60	150%
Tennessee	281	258	339	410	83%
Texas	1,260	1,040	1,425	1,120	127%
Utah	96	95	144	80	180%
Vermont	66	55	71	67	106%
Virginia	456	408	519	595	87%
Washington	374	270	511	380	134%
West Virginia	133	152	183	262	70%
Wisconsin	626	547	734	700	105%
Wyoming	63	55	83	60	138%
TOTALS	22,795	21,706	29,937	24,063	124%
U. S. Possessions	33	25	36	40	90%
Canada	451	385	670	335	200%
Foreign	446	249	551	565	98%
GRAND TOTALS	23,790	22,365	31,194	25,003	125%

Members and Student Members, NCTM May 1, 1960

	Members	*Student Members*	*Per Cent Student Members*
Alabama	221	19	9%
Alaska	12	2	17%
Arizona	265	130	49%
Arkansas	155	2	1%
California	1,493	149	10%
Colorado	319	47	15%
Connecticut	357	20	6%
Delaware	83	3	4%
District of Columbia	208	6	3%
Florida	547	105	19%
Georgia	272	26	10%
Hawaii	99	46	46%
Idaho	28	2	7%
Illinois	1,622	251	15%
Indiana	753	128	17%
Iowa	390	64	16%
Kansas	425	30	7%
Kentucky	185	6	3%
Louisiana	351	46	13%
Maine	111	10	9%
Maryland	383	29	8%
Massachusetts	640	44	7%
Michigan	1,248	414	33%
Minnesota	690	139	20%
Mississippi	159	11	7%
Missouri	423	44	10%
Montana	120	18	15%
Nebraska	208	32	15%
Nevada	29	0	0%
New Hampshire	102	3	3%
New Jersey	835	127	15%
New Mexico	112	9	8%
New York	2,128	273	13%
North Carolina	285	11	4%
North Dakota	101	40	40%
Ohio	1,091	124	11%
Oklahoma	327	8	2%
Oregon	298	71	24%
Pennsylvania	1,437	301	21%
Rhode Island	109	23	21%
South Carolina	144	4	3%
South Dakota	61	14	23%
Tennessee	258	19	7%
Texas	1,040	108	10%
Utah	95	6	6%
Vermont	55	0	0%
Virginia	408	13	3%
Washington	270	15	6%
West Virginia	152	11	7%
Wisconsin	547	101	18%
Wyoming	55	0	0%
TOTALS	21,706	3,104	14%
Canada	385	54	14%
Foreign	249	2	1%
U. S. Possessions	25	0	0%
GRAND TOTALS	22,365	3,160	14%

You will be interested to note:

41 states and territories are well over the current 25,000 goals.
8 states and territories need fewer than 25 members to reach their goals.
5 states have not been so successful and doubtless need your help. Can any of you do more to help the leaders in these states?
24 states have had continuous growth during the past year.
10 states are already well *over their 40,000 goals.*

We have *3160 student members.* We are pleased about this, for we are convinced this interest will continue with these young people as our future teachers of America.

Future Plans

My earlier remarks have presupposed a continuation of the work of a Membership Committee. This may not be the case. The new President and the newly organized Board of Directors may have other directives to issue.

If the Committee does continue, the organization of its work and certain details of procedure currently used may be somewhat changed. The nature of such changes has not yet been determined, but they will be in keeping with the reorganization plans recently adopted by the Council.

The Membership Committee again extends to each of you its sincere thanks for your outstanding co-operation and helpfulness. Please accept our best wishes for your greatest professional success at all times.

Pearl Bond
Marian C. Cliffe
Mary Lee Foster
Janet Height
Harold J. Hunt
Lucille Houston
Florence Ingham
Mildred Keiffer
Faith Novinger
Bess Patton
Myrl H. Ahrendt and
Eugene P. Smith, Ex officio members
Mary C. Rogers, Chairman

Report of the Nominating Committee

The following named persons have been nominated for the indicated offices in The National Council of Teachers of Mathematics. Biographies and photographs of these nominees will be published in the January issue of The Mathematics Teacher.

Vice President, College Level

Bruce Meserve, New Jersey
Myron Rosskopf, New York

Vice President, Junior High School Level

Helen Garstens, Maryland
Mildred Keiffer, Ohio

Directors

Max Beberman, Illinois
Lurnice Begnaud, Louisiana
John A. Brown, Maryland
W. T. Guy, Texas
Julius Hlavaty, New York
Rachel Keniston, California

Ballots will be mailed on or before February 15, 1961 from the Washington office to members of record as of that date. Ballots returned and postmarked not later than March 15, 1961 will be counted.

Respectfully submitted,
Oscar F. Schaaf, Chairman

• NOTES FROM THE WASHINGTON OFFICE

Edited by M. H. Ahrendt, Executive Secretary, NCTM, Washington, D. C.

Registrations at NCTM meetings

Below are registration reports for meetings of the Council held during 1959–1960. The Nineteenth Summer Meeting and the Thirty-eighth Annual Meeting were the largest meetings of their type ever held by the Council, while the Joint Meeting with the NEA was the second largest of its particular kind.

Of special significance was the Joint Meeting with the Mathematical Association of America. This meeting was well attended by persons from higher education, and it illustrated the co-operative approach being used by persons at all levels of instruction to improve the teaching of mathematics.

Registrations at the Nineteenth Summer Meeting

The National Council of Teachers of Mathematics, University of Michigan, Ann Arbor, Michigan, August 16–19, 1959

Alabama	5	Missouri	12
Alaska	1	Montana	1
Arizona	4	Nebraska	5
Arkansas	1	New Hampshire	1
California	23	New Jersey	17
Colorado	4	New York	39
Connecticut	5	North Carolina	5
Delaware	1	Ohio	72
District of Columbia	10	Oklahoma	1
Florida	4	Oregon	2
Georgia	6	Pennsylvania	35
Hawaii	2	South Dakota	1
Illinois	96	Tennessee	2
Indiana	66	Texas	11
Iowa	19	Virginia	7
Kansas	10	Washington	1
Kentucky	2	West Virginia	11
Louisiana	7	Wisconsin	42
Maryland	9	Canada	36
Massachusetts	17	Foreign	14
Michigan	305		
Minnesota	10		
Mississippi	1	TOTAL	923

Registrations at the Thirty-eighth Annual Meeting

The National Council of Teachers of Mathematics, Buffalo, New York, April 20–23, 1960.

Alabama	3	New Jersey	48
Arizona	1	New Mexico	1
Arkansas	2	New York	1,005
California	22	North Carolina	2
Colorado	7	North Dakota	2
Connecticut	47	Ohio	112
Delaware	5	Oklahoma	2
District of Columbia	27	Oregon	2
Florida	13	Pennsylvania	169
Georgia	6	Rhode Island	6
Illinois	105	South Carolina	1
Indiana	38	South Dakota	1
Iowa	13	Tennessee	7
Kansas	2	Texas	16
Kentucky	2	Utah	2
Louisiana	7	Vermont	2
Maine	5	Virginia	25
Maryland	37	Washington	2
Massachusetts	71	West Virginia	3
Michigan	83	Wisconsin	16
Minnesota	30	Canada	62
Mississippi	3	Puerto Rico	1
Missouri	19	Foreign	2
Nebraska	9		
Nevada	1		
New Hampshire	9	Total	2,056

Registrations at Joint Meeting with the Mathematical Association of America

The National Council of Teachers of Mathematics, Chicago, Illinois, January 30, 1960

Alabama	2	New Jersey	15
Alaska	1	New Mexico	2
Arkansas	1	New York	38
California	16	North Carolina	7
Colorado	3	North Dakota	1
Connecticut	8	Ohio	49
Delaware	2	Oklahoma	8
District of Columbia	11	Oregon	2
Florida	8	Pennsylvania	21
Illinois	503	South Carolina	1
Indiana	102	South Dakota	5
Iowa	28	Tennessee	9
Kansas	10	Texas	6
Kentucky	14	Utah	1
Louisiana	7	Virginia	6
Maryland	7	Washington	5
Massachusetts	18	West Virginia	1
Michigan	52	Wisconsin	106
Minnesota	31	Canada	3
Missouri	27	Puerto Rico	1
Montana	3		
Nebraska	4		
New Hampshire	1	TOTAL	1,146

Registrations at Joint Meeting with the National Education Association

The National Council of Teachers of Mathematics, Los Angeles, California, June 29, 1960

Alabama	2
Alaska	1
Arizona	1
Arkansas	2
California	192
Colorado	2
Connecticut	1
District of Columbia	1
Florida	2
Georgia	2
Idaho	2
Illinois	2
Indiana	1
Iowa	6
Kansas	6
Kentucky	3
Louisiana	3
Maryland	1
Massachusetts	1
Michigan	8
Minnesota	3
Missouri	6
Nebraska	2
New Hampshire	1
New Jersey	1
New Mexico	3
New York	3
North Carolina	1
North Dakota	2
Ohio	14
Oklahoma	2
Oregon	2
Pennsylvania	3
South Dakota	2
Tennessee	3
Texas	5
Utah	1
Virginia	3
Washington	2
West Virginia	2
Wisconsin	1
Wyoming	1
TOTAL	302

Your professional dates

The information below gives the name, date, and place of meeting with the name and address of the person to whom you may write for further information. For information about other meetings, see the previous issues of THE MATHEMATICS TEACHER. Announcements for this column should be sent at least ten weeks early to the Executive Secretary, National Council of Teachers of Mathematics, 1201 Sixteenth Street, N. W., Washington 6, D. C.

NCTM convention dates

NINETEENTH CHRISTMAS MEETING

December 27–30, 1960
Arizona State University, Tempe, Arizona
Lehi Smith, Arizona State University, Tempe, Arizona

THIRTY-NINTH ANNUAL MEETING

April 5–8, 1961
Conrad Hilton Hotel, Chicago, Illinois
Hobert Sisler, Morton High School West, 2400 Home Avenue, Berwyn, Illinois

JOINT MEETING WITH NEA

June 28, 1961
Atlantic City, New Jersey
M. H. Ahrendt, 1201 Sixteenth Street, N. W., Washington 6, D. C.

TWENTY-FIRST SUMMER MEETING

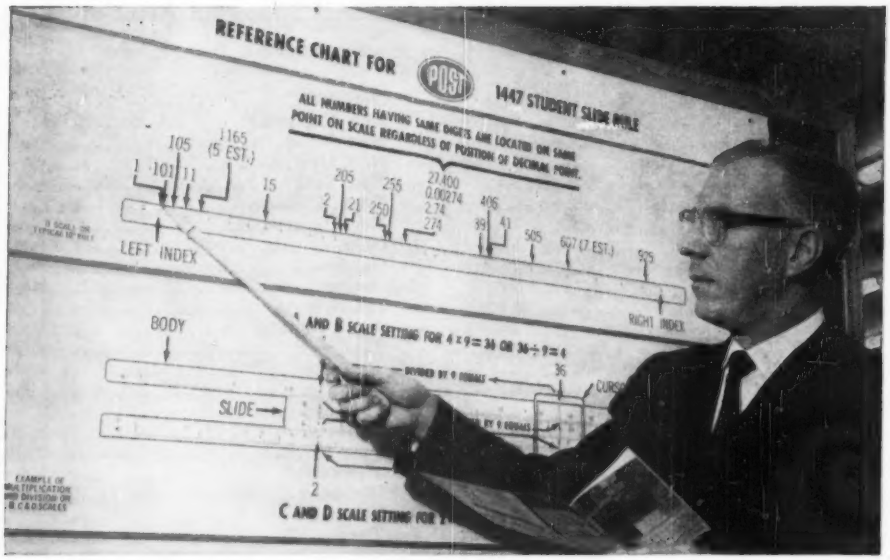
August 21–23, 1961
University of Toronto, Toronto, Canada
Father John C. Egsgard, St. Michael's College School, Toronto, Canada

Other professional dates

Men's Mathematics Club of Chicago and Metropolitan Area
November 18, 1960
YMCA Hotel, 826 South Wabash Avenue, Chicago, Illinois
Vernon R. Kent, 1510 South Sixth Avenue, Maywood, Illinois

Northern Section, California Mathematics Council
December 9–11, 1960
Asilomar Fall Conference
Mike Donahoe, P. O. Box 1385, Carmel, California

Men's Mathematics Club of Chicago and Metropolitan Area
December 16, 1960
YMCA Hotel, 826 South Wabash Avenue, Chicago, Illinois
Vernon R. Kent, 1510 South Sixth Avenue, Maywood, Illinois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ALL NUMBERS HAVING SAME DIGITS ARE LOCATED ON SAME POINT ON SCALE REGARDLESS OF POSITION OF DECIMAL POINT.
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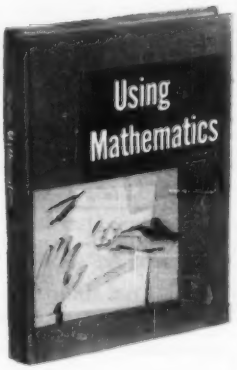

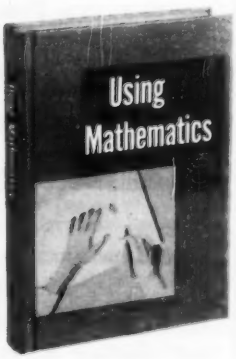

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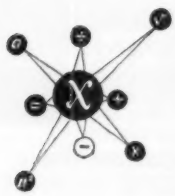

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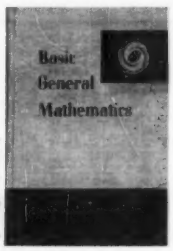

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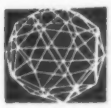

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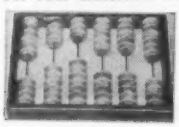

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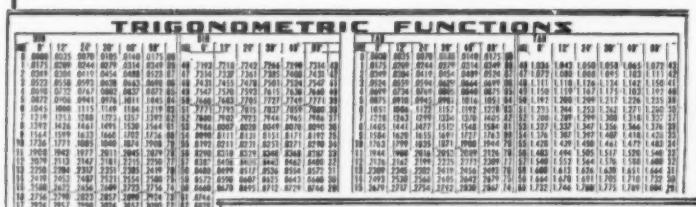

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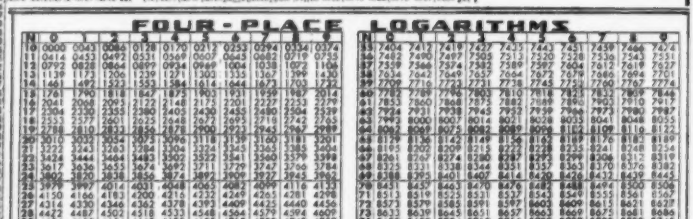

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